COMBINATORICS OF IDEALS – SELECTIVITY VERSUS DENSITY

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ABSTRACT. This paper is devoted to combinatorial properties of ideals on countable sets. By an old result of Mathias, two such properties, selectivity and density, in the case of definable ideals exclude each other. The main purpose of this note is to measure the "distance" between them via ultrafilter topologies of Louveau and countable diagonalizations of Laflamme. In particular, some new characterizations of selective and selective-like ideals are obtained.

1. INTRODUCTION

This note is mostly concerned with the following two combinatorial properties of ideals. An ideal $\mathcal{I}$ on a countable set D is

- *dense* (or *tall*) if every infinite subset of D contains an infinite subset in $\mathcal{I}$,
- *selective* if for every partition $\{A_n : n \in \omega\}$ of D such that no finite union of elements of the partition is in the dual filter of $\mathcal{I}$ there is a selector in $\mathcal{I}^+$, i.e., a set $S \in \mathcal{I}^+$ such that $|S \cap A_n| \leq 1$ for every $n \in \omega$. (Note that the assumption that no finite union of elements of the partition is in the dual filter of $\mathcal{I}$ is clearly a necessary condition for the existence of its selector in $\mathcal{I}^+$.)

While density is a rather common property of ideals, the list of presently known examples of selective ideals is apparently short. It consists of countably generated ideals, ideals generated by AD families of subsets of ω, ideals of the form

$$I_K(x,(x_n)) = \{M \subseteq \omega : \ x \notin \overline{\{x_n : \ n \in M\}}\}$$

(where K is a topological space with some suitable properties, $x \in K$ and (x_n) is a sequence of elements of $K \setminus \{x\}$ accumulating to x) and the maximal ideals whose duals are Ramsey ultrafilters. Moreover, Mathias [13] proved that if $\mathcal{I}$ is an analytic (or coanalytic) selective ideal on ω, then $\mathcal{I}$ is not dense.

The main result of this note (Theorem 2.3) shows that selectivity of a definable ideal $\mathcal{I}$ is equivalent to $\mathcal{I}$ being nowhere dense in *every*

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so-called ultrafilter topology on $[\omega]^\omega$ (associated with any ultrafilter extending the dual filter of $\mathcal{I}$) studied earlier by Louveau [10], Todorčević [21] and others. On the other hand, the property of $\mathcal{I}$ being *not* dense is equivalent to $\mathcal{I}$ being nowhere dense in *at least one* of such topologies.

Complementing an earlier result of Todorčević we also show (Proposition 3.7) that a definable ideal $\mathcal{I}$ is selective if and only if it is ω-diagonalizable (in the sense of Laflamme [9]) by elements of $\mathscr{U}$ for every ultrafilter $\mathscr{U}$ extending the dual filter of $\mathcal{I}$. On the other hand, it is easy to see (Proposition 3.4) that $\mathcal{I}$ is *not* dense if and only if it is ω-diagonalizable by elements of at least one filter.

The last part of the paper is devoted to other selective-like properties of ideals with the particular emphasis on their connections with selectivity and density.

1.1. **Basic definitions and notation.** An *ideal* on a countable (infinite) set D is a collection $\mathcal{I}$ of subsets of D which is closed under taking subsets and finite unions. We always assume that $\mathcal{I} \neq \mathcal{P}(D)$ and $\mathcal{I}$ contains all finite subsets of D. A *filter* on a countable set D is a collection $\mathcal{F}$ of subsets of D which is closed under taking supersets and finite intersections. We always assume that $\mathcal{F} \neq \mathcal{P}(D)$ and $\mathcal{F}$ contains all cofinite subsets of D (in particular, no finite subset of D is in $\mathcal{F}$).

If $\mathcal{I}$ is an ideal on D, then $\mathcal{I}^* = \{B \subseteq D : B^c \in \mathcal{I}\}$ is the *dual filter* and $\mathcal{I}^+ = \mathcal{I}^c = \mathcal{P}(D) \setminus \mathcal{I}$ is the *associated coideal* of $\mathcal{I}$ consisting of $\mathcal{I}$*-positive* sets. For any set $A \in \mathcal{I}^+$ by $\mathcal{I}|A$ we denote the ideal of subsets of A defined by $\mathcal{I}|A = \mathcal{I} \cap \mathcal{P}(A)$.

An ideal $\mathcal{I}$ on D is *generated by a family* $\mathcal{A} \subseteq \mathcal{P}(D)$ if it is the smallest ideal on D containing all elements of $\mathcal{A}$ and all finite subsets of D. A *base* of an ideal $\mathcal{I}$ is a subset $\mathscr{B}$ of $\mathcal{I}$ such that each element of $\mathcal{I}$ is a subset of an element of $\mathscr{B}$.

We use standard set-theoretic notation. In particular, by $[\omega]^{<\omega}$ (respectively: $[\omega]^\omega$) we denote the collection of all finite (respectively: infinite) subsets of ω, the set of natural numbers. If D is a countable set, then the space 2^D of all functions $f : D \to \{0, 1\}$, equipped with the product topology, is the Cantor set. By identifying subsets of D with their characteristic functions we also treat the power set $\mathcal{P}(D)$ as the space 2^D. Descriptive set theoretic notions concerning ideals on D (like being $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, Borel, analytic or coanalytic; see [5]) always refer to the Cantor set topology on $\mathcal{P}(D)$.

1.2. **Examples of selective ideals.** While density is a rather common property of ideals, the following list apparently exhausts all the presently known examples of selective ideals.

Example 1.1. Any countably generated ideal is selective (see e.g. [22]) (an ideal is countably generated if it is generated by a countable family; note that any countably generated ideal $\mathcal{I}$ has a countable base). Such an ideal is clearly $\mathbf{\Sigma^0_2}$ and not dense since if $\mathscr{B}$ is a countable base of $\mathcal{I}$, then no infinite set almost disjoint from every element of $\mathscr{B}$ is in $\mathcal{I}$.

Example 1.2. (Mathias [13]) The ideal generated by an infinite almost-disjoint family $\mathcal{A}$ (shortly: an AD family) of subsets of ω is selective. It is easy to see that this ideal is dense if and only if $\mathcal{A}$ is a maximal infinite almost-disjoint family (shortly: a MAD family). If $\mathcal{A}$ is analytic, so is the ideal generated by $\mathcal{A}$. Moreover, Calbrix [2] constructed ideals of this form (i.e., generated by AD families) of an arbitrarily high Borel complexity. More precisely, given a $\mathbf{\Sigma^0_\alpha}$-complete, $\alpha > 1$, (respectively: $\mathbf{\Pi^0_\alpha}$-complete, $\alpha > 2$) subset A of 2^ω, the ideal $\mathcal{I}_A$ on $2^{<\omega}$ generated by the family $\{B_x : x \in A\}$ of all branches of the tree $2^{<\omega}$ determined by elements of A, i.e., sets of the form

$$B_x = \{s \in 2^{<\omega} : \forall i < \mathrm{lh}(s)\ s(i) = x(i)\}$$

for $x \in A$ (cf. [11]) is a selective ideal, which is $\mathbf{\Sigma^0_\alpha}$-complete (respectively: $\mathbf{\Pi^0_\alpha}$-complete).

Example 1.3. (Todorčević [20, Section 12] and [21]) The ideal topologically represented (in the sense of Todorčević [20]) by a sequence of continuous real-valued functions (f_n) on a Polish space X accumulating pointwise to a continuous function f on X, i.e., the ideal

$$\mathcal{I} = \{M \subseteq \omega :\ f \notin \overline{\{f_n :\ n \in M\}}\}$$

where, moreover, the pointwise closure of (f_n) consists of Baire Class 1 functions defined on X, is selective and analytic. This ideal is not dense since by the Bourgain-Fremlin-Talagrand theorem (see [20, Section 13]), (f_n) has a subsequence converging to f. Non-density of the ideal $\mathcal{I}$ follows also, by a theorem of Mathias (cf. Example 1.5), directly from the fact that $\mathcal{I}$ is analytic and selective. Note that *every* analytic ideal on ω is topologically represented by a sequence of continuous real-valued functions (f_n) on a Polish space X accumulating pointwise to a continuous function f (see [21, Lemma 6.53]). Hence selectivity (and consequently: non-density) of the ideal $\mathcal{I}$ required imposing on a sequence (f_n) some additional restrictions what we did above.

More generally, let $\mathcal{I}_K(x,(x_n))$ denote the ideal *represented by* K, a regular topological space, in the following way:

$$\mathcal{I}_K(x,(x_n)) = \{M \subseteq \omega :\ x \notin \overline{\{x_n :\ n \in M\}}\},$$

for a point $x \in K$ and a sequence (x_n) of elements of $K \setminus \{x\}$ accumulating to x (see [21, Section 6.7]). Then the ideal $\mathcal{I} = \mathcal{I}_K(x,(x_n))$ is selective, if either K is a countably tight compact space in which case we additionally assume that $\mathcal{I}|A$ has the Baire property in $\mathcal{P}(A)$ for every $A \in \mathcal{I}^+$ (for details see [21, Theorem 7.47]) or K is a countably tight sequentially compact space (for details see [21, Theorem 7.48]). In particular, the latter is the case when K is a compact set of Baire Class 1 functions defined on some Polish space X (i.e., K is so-called

Rosenthal compactum). By a result of Marciszewski and Pol (see [12, Lemma 4.4]), if $\mathcal{A}$ is an analytic AD family, then the ideal generated by $\mathcal{A}$ is represented by a Rosenthal compactum.

It is easy to see that the ideal $\mathcal{I}_K(x,(x_n))$ is dense if and only if no subsequence of (x_n) converges to x.

Example 1.4. A maximal ideal is selective if and only if its dual filter is a Ramsey ultrafilter (see [1, Theorem 4.5.2]). Since a Ramsey ultrafilter is a P-point, its existence cannot be proved in ZFC. Clearly, if a maximal ideal is selective, then it is dense.

1.3. **Examples of ideals which are not selective.** As indicated above, selectivity is a rather rare property. The following list provides examples of large classes of ideals which are not selective.

Example 1.5. (Mathias [13]) No analytic (or coanalytic) dense ideal is selective.

Since this theorem of Mathias is the main motivation for the results of this note, for the sake of completeness we sketch one of its proofs.

Call a family $\mathcal{C} \subseteq [\omega]^\omega$ *$\mathcal{I}$-Ramsey*, if for all $A \in \mathcal{I}^+$ there is $B \subseteq A$ in $\mathcal{I}^+$ such that $[B]^\omega \cap \mathcal{C} = \emptyset$ or $[B]^\omega \subseteq \mathcal{C}$.

Mathias [13] showed that analytic sets are $\mathcal{I}$-Ramsey for every selective ideal $\mathcal{I}$.

Hence if an ideal $\mathcal{I}$ is analytic (or coanalytic) and selective, then it is itself $\mathcal{I}$-Ramsey so taking $A = \omega$ we get $B \in \mathcal{I}^+$ such that $[B]^\omega \subseteq \mathcal{I}^+$ which shows that $\mathcal{I}$ is not dense.

A well-known corollary to the above theorem is the fact that no infinite MAD family of subsets of ω is analytic (otherwise the ideal generated by such a family would be analytic, selective and dense).

Example 1.6. (Zakrzewski [22]) No analytic non-countably generated P-ideal is selective (an ideal $\mathcal{I}$ is a P-ideal if for any sequence $X_n \in \mathcal{I}^*$, $n \in \omega$, there is an $X \in \mathcal{I}^*$ such that $X \subseteq^* X_n$ for all $n \in \omega$, i.e., $X \setminus X_n$ is finite for all $n \in \omega$). This is in contrast with the obvious fact that every selective ideal is a *P^+-ideal*, i.e., for any decreasing sequence $X_n \in \mathcal{I}^+$, $n \in \omega$, there is an $X \in \mathcal{I}^+$ such that $X \subseteq^* X_n$ for all $n \in \omega$ (see [4]). In fact it is well known (and easy to check) that an ideal $\mathcal{I}$ is selective if and only if it is both P^+ and *Q^+-ideal* where the latter means that for every set $X \in \mathcal{I}^+$ and every partition $\{F_n : n < \omega\}$ of X into finite sets there is a selector in $\mathcal{I}^+$ (see [4]).

Example 1.7. The ideal topologically represented (in the sense of Sabok and Zapletal [17]) by a σ-ideal $\mathcal{J}$ (containing all singletons) on an uncountable Polish space X, i.e., the ideal on a countable dense subset D of X defined by the formula

$$\mathcal{I}_{\mathcal{J}} = \{A \subseteq D : \overline{A} \in \mathcal{J}\}$$

is not selective.

More precisely, $\mathcal{I} = \mathcal{I}_{\mathcal{J}}$ is not a P^+-ideal. To see this, first note that there is a point $x_0 \in X$ such that no open ball $B(x_0, r)$, $r > 0$, is in $\mathcal{J}$. Letting $X_n = D \cap B(x_0, \frac{1}{n+1})$, $n \in \omega$, we obtain a decreasing sequence of $\mathcal{I}$-positive subsets. Suppose that $X \subseteq D$ is such that $X \subseteq^* X_n$ for all $n \in \omega$. Then $\overline{X} = X \cup \{x_0\} \in \mathcal{J}$, hence $X \in \mathcal{I}$.

On the other hand, by [17, Proposition 4.3], $\mathcal{I}$ is not only a Q^+ but even a *weakly selective* ideal, i.e., for every set $X \in \mathcal{I}^+$ and every partition $\{F_n : n < \omega\}$ of X into sets from $\mathcal{I}$ there is a selector in $\mathcal{I}^+$.

It is easy to see that $\mathcal{I}$ is dense and in many cases of interest it is analytic, and consequently, by [8, Theorem 1.4], $\mathbf{\Pi^0_3}$-complete. In particular, this is the case for the nowhere dense ideal *nwd* on $\mathbb{Q}$ (consisting of nowhere dense subsets of $\mathbb{Q}$).

Example 1.8. There are non-selective ideals of arbitrarily high Borel complexities. More precisely, if $\alpha > 1$ (respectively: $\alpha > 2$), then there is a non-selective ideal which is $\mathbf{\Sigma^0_\alpha}$-complete (respectively: $\mathbf{\Pi^0_\alpha}$-complete).

To see this, let $\mathcal{I}_1$ be an arbitrary non-selective, $\mathbf{\Sigma^0_2}$ ideal on ω (e.g., $\mathcal{I}_1 = \{M \subseteq \omega : \sum_{n \in M} \frac{1}{n+1} < \infty\}$). Fix an $\alpha > 1$ (respectively: $\alpha > 2$) and let $\mathcal{I}_2$ be an ideal on $2^{<\omega}$ of the form $\mathcal{I}_A$ (see Example 1.5) where A is a $\mathbf{\Sigma^0_\alpha}$-complete (respectively: $\mathbf{\Pi^0_\alpha}$-complete) subset of 2^ω. Finally, let $\mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2$, i.e., the ideal $\mathcal{I}$ is defined on $D = \{0\} \times \omega \cup \{1\} \times 2^{<\omega}$ by declaring that for $A \subseteq D$ we have

$$A \in \mathcal{I} \;\Leftrightarrow\; \{n \in \omega : (0, n) \in A\} \in \mathcal{I}_1 \;\wedge\; \{s \in 2^{<\omega} : (1, s) \in A\} \in \mathcal{I}_2.$$

It is easy to see that since $\mathcal{I}_1$ is non-selective, so is $\mathcal{I}$ and the descriptive complexity of $\mathcal{I}$ is precisely that of $\mathcal{I}_2$, i.e., that of A (cf. Example 1.5).

1.4. **Trees and Grigorieff's characterization of selectivity.** Our notation and terminology concerning trees agrees with [18] and is also close to [21].

By $\sqsubseteq$ we denote the initial segment relation on $\mathcal{P}(\omega)$, i.e., for $s \in [\omega]^{<\omega}$ and $A \subseteq \omega$ we have

$$s \sqsubseteq A \quad \Leftrightarrow \quad \forall i \leq \max(s)\ (i \in s \Leftrightarrow i \in A).$$

By a *tree* we mean a non-empty set $T \subseteq [\omega]^{<\omega}$ such that $s \in T$ and $t \sqsubseteq s$ imply $t \in T$.

If T is a tree and $s \in T$ then we denote by $succ_T(s)$ the set $\{n > \max(s) : \; s \cup \{n\} \in T\}$. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we say that T is an *$\mathcal{A}$-tree* if $succ_T(s) \in \mathcal{A}$ for every $s \in T$.

We say that a tree T is a *centered $\mathcal{I}^+$-tree* (or a *strong $\mathcal{I}^+$-tree*, cf. [3]) if for any finite set $\{s_i : i < n\} \subseteq T$ we have

$$\bigcap\{succ_T(s_i) : \; i < n\} \in \mathcal{I}^+.$$

A set $A \in [\omega]^\omega$ is *a branch* of a tree T if $s \sqsubseteq A$ implies $s \in T$ for every $s \in [\omega]^{<\omega}$. By $[T]$ we denote the set of all branches of T.

The following important characterization of selectivity is due to Grigorieff.

Theorem 1.9. (Grigorieff [3]) An ideal $\mathcal{I}$ on ω is selective if and only if every centered $\mathcal{I}^+$-tree has a branch in $\mathcal{I}^+$.

1.5. **The $\mathscr{U}$-topology on $[\omega]^\omega$.** Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω. The following basic facts concerning the so-called $\mathscr{U}$-topology are taken from [21] (cf. [18]).

For $s \in [\omega]^{<\omega}$ and a $\mathscr{U}$-tree T such that $\max(s) < \min(\bigcup T)$ we let

$$[s, T] = \{A \in [\omega]^\omega : \ s \sqsubseteq A \ \wedge \ A \setminus s \in [T]\}.$$

We say that a subset $\mathcal{G}$ of $[\omega]^\omega$ is *$\mathscr{U}$-open* if for every $B \in \mathcal{G}$ there are s and T as above such that $B \in [s, T]$ and $[s, T] \subseteq \mathcal{G}$.

Proposition 1.10. (Louveau [10]; see [21, Lemma 7.36]) The family of all $\mathscr{U}$-open sets is a topology (called *the $\mathscr{U}$-topology $\tau_{\mathscr{U}}$*) on $[\omega]^\omega$ (with the basis $\mathscr{E}_{\mathscr{U}}$ consisting of sets of the form $[s, T]$). The $\mathscr{U}$-topology extends the Polish topology on $[\omega]^\omega$ inherited from $\mathcal{P}(\omega)$ identified with the Cantor set 2^ω.

Let $\mathcal{X} \subseteq [\omega]^\omega$. We say that $\mathcal{X}$ is

- *completely $\mathscr{U}$-Ramsey* if for every basic set $[s, T] \in \mathscr{E}_{\mathscr{U}}$ there is a $\mathscr{U}$-tree $T' \subseteq T$ such that $[s, T'] \subseteq \mathcal{X}$ or $[s, T'] \subseteq \mathcal{X}^c$.
- *completely $\mathscr{U}$-Ramsey null* if for every basic set $[s, T] \in \mathscr{E}_{\mathscr{U}}$ there is a $\mathscr{U}$-tree $T' \subseteq T$ such that $[s, T'] \subseteq \mathcal{X}^c$.

Theorem 1.11. (Louveau [10]; "Ultra-Ellentuck Theorem", cf. [21]) A set $\mathcal{X} \subseteq [\omega]^\omega$ is completely $\mathscr{U}$-Ramsey if and only if it has the Baire property in the topology $\tau_{\mathscr{U}}$. Moreover, $\mathcal{X}$ is completely $\mathscr{U}$-Ramsey null if and only if it is meager in $\tau_{\mathscr{U}}$.

Corollary 1.12. If $\mathcal{X} \subseteq [\omega]^\omega$ is analytic (or coanalytic), then $\mathcal{X}$ is $\tau_{\mathscr{U}}$-dense in $[\omega]^\omega$ if and only if $\mathcal{X}^c$ is $\tau_{\mathscr{U}}$-nowhere dense.

2. Selectivity versus density via ultrafilter topologies

The results of this section characterize selectivity and density of ideals in terms of ultrafilter topologies.

Theorem 2.1. Let $\mathcal{I}$ be an ideal on ω. Then:

(i) $\mathcal{I}$ is selective if and only if $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$ (equivalently: $\mathcal{I}^* \subseteq \mathscr{U}$).

(ii) $\mathcal{I}$ is dense if and only if $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every nonprincipal ultrafilter $\mathscr{U}$.

(iii) $\mathcal{I}$ is dense if and only if $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.

Proof. (i). Assume that $\mathcal{I}$ is selective. Let $[s,T] \in \mathscr{E}_{\mathscr{U}}$ be a basic open set in the $\mathscr{U}$-topology related to an ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.

Then T being a $\mathscr{U}$-tree is also a centered $\mathcal{I}^+$-tree. Hence, by Theorem 1.9 (the Grigorieff's characterization), there is $A \in [T] \cap \mathcal{I}^+$. Then $s \cup A \in [s,T] \cap \mathcal{I}^+$ which shows that $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$.

Now assume that $\mathcal{I}^+$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Let T be a centered $\mathcal{I}^+$-tree; by Theorem 1.9, to prove that $\mathcal{I}$ is selective it is enough to show that $[T] \cap \mathcal{I}^+ \neq \emptyset$.

Extend the filter $\mathcal{I}^*$ to an ultrafilter $\mathscr{U}$ such that $succ_T(s) \in \mathscr{U}$ for each $s \in T$. Then, T being a $\mathscr{U}$-tree, $[T] \in \tau_{\mathscr{U}}$. Since $\mathcal{I}^+$ is $\tau_{\mathscr{U}}$-dense, $[T] \cap \mathcal{I}^+ \neq \emptyset$.

(ii) and (iii). Assume that $\mathcal{I}$ is dense. Let $[s,T] \in \mathscr{E}_{\mathscr{U}}$ be a basic open set in the $\mathscr{U}$-topology related to a nonprincipal ultrafilter $\mathscr{U}$.

By shrinking T, if necessary, we assume with no loss of generality that $[B]^\omega \subseteq [T]$ for every $B \in [T]$. More precisely, let

$$T^d = \{s \in T : \ \forall t \subseteq s \quad t \in T\}.$$

Then T^d is a $\mathscr{U}$-tree, $T^d \subseteq T$ and $[A]^\omega \subseteq [T^d]$ for any $A \in [T^d]$ (for details see [18, Lemma 1 of 2.2]).

Let $B \in [T]$; since $\mathcal{I}$ is dense, there is $A \in [B]^\omega \cap \mathcal{I}$. Then $A \in [T] \cap \mathcal{I}$ so $s \cup A \in [s,T] \cap \mathcal{I}$ which shows that $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$.

Now assume that $\mathcal{I}$ is dense in $\tau_{\mathscr{U}}$ for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Let $B \in \mathcal{I}^+$; our aim is to show that $[B]^\omega \cap \mathcal{I} \neq \emptyset$.

Extend the filter $\mathcal{I}^*$ to an ultrafilter $\mathscr{U}$ such that $B \in \mathscr{U}$. Let $T = [B]^{<\omega}$.

Then, T being a $\mathscr{U}$-tree, $[T] \in \tau_{\mathscr{U}}$. Since $\mathcal{I}$ is $\tau_{\mathscr{U}}$-dense, there is $A \in [T] \cap \mathcal{I} \subseteq [B]^\omega \cap \mathcal{I}$ showing that $[B]^\omega \cap \mathcal{I} \neq \emptyset$. □

By the theorem of Mathias recalled in Introduction (see Example 1.5), selectivity and density in the case of definable ideals exclude each other. The following corollary describes the distance between these properties in the language of $\mathscr{U}$-topologies. We precede it with a general lemma.

Lemma 2.2. If $\mathscr{U}$ is a nonprincipal ultrafilter on ω and $\mathcal{I}$ is an ideal on ω, then $\mathcal{I}$ is not dense in $\tau_{\mathscr{U}}$ if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense.

Proof. Since it is clearly enough to prove the "only if" direction, assume that $\mathcal{I}$ is not dense in $\tau_{\mathscr{U}}$. This means that there is a basic $\mathscr{U}$-open set $[s_1,T_1] \in \mathscr{E}_{\mathscr{U}}$ disjoint from $\mathcal{I}$. Let $[s_2,T_2] \in \mathscr{E}_{\mathscr{U}}$ be an arbitrary basic $\mathscr{U}$-open set. Then $T = T_1 \cap T_2$ is a $\mathscr{U}$-tree (see [18, Lemma 2 of 3.2]) and $[s_2,T] \subseteq [s_2,T_2]$. Moreover, $[s_2,T] \cap \mathcal{I} = \emptyset$. Indeed, if there was $A \in [s_2,T] \cap \mathcal{I}$, then we would have

$$(A \setminus s_2) \cup s_1 \in [s_1,T] \cap \mathcal{I} \subseteq [s_1,T_1] \cap \mathcal{I},$$

contradicting the assumption that $[s_1,T_1] \cap \mathcal{I} = \emptyset$. □

Theorem 2.3. If $\mathcal{I}$ is an analytic (or coanalytic) ideal on ω, then:

(i) $\mathcal{I}$ is selective if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.

(ii) $\mathcal{I}$ is dense if and only if $\mathcal{I}$ contains a $\tau_{\mathscr{U}}$-dense open subset for every nonprincipal ultrafilter $\mathscr{U}$. Moreover, $\mathcal{I}$ is not dense if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for a certain nonprincipal ultrafilter $\mathscr{U}$.

(iii) $\mathcal{I}$ is dense if and only if $\mathcal{I}$ contains a $\tau_{\mathscr{U}}$-dense open subset for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. Moreover, $\mathcal{I}$ is not dense if and only if $\mathcal{I}$ is $\tau_{\mathscr{U}}$-nowhere dense for a certain ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$.

Proof. Point (i) and the first parts of points (ii) and (iii) follow immediately from the respective points of Theorem 2.1 and Corollary 1.12 of Theorem 1.11 (the ultra-Ellentuck theorem of Louveau). The "moreover" parts of (ii) and (iii) are consequences of Lemma 2.2. □

Mathias [13] characterized selective ultrafilters as exactly those ultrafilters which have non-empty intersection with every dense analytic ideal. The following result generalizes this property of selective ultrafilters.

Theorem 2.4. Let $\mathcal{I}$ be an ideal on ω. If $\mathscr{U}$ is a nonprincipal selective ultrafilter on ω and $\mathcal{I}$ is a dense ideal on ω having the Baire property in the topology $\tau_{\mathscr{U}}$, then $\mathcal{I} \cap \mathscr{U} \neq \emptyset$.

Proof. Since, by Theorem 1.11, $\mathcal{I}$ is completely $\mathscr{U}$-Ramsey, there is a $\mathscr{U}$-tree T such that $[T] \subseteq \mathcal{I}$ or $[T] \subseteq \mathcal{I}^+$. By Theorem 2.1, $\mathcal{I}$ being dense is $\tau_{\mathscr{U}}$-dense. It follows that $[T] \subseteq \mathcal{I}$.

On the other hand, $\mathscr{U}$ being selective and T being a (centered) $\mathscr{U}$-tree, Theorem 1.9 implies that $[T] \cap \mathscr{U} \neq \emptyset$. Let $A \in [T] \cap \mathscr{U}$. Then $A \in \mathcal{I} \cap \mathscr{U}$. □

3. Selectivity versus density via ω-diagonalizations

Following Laflamme [9] we say that an ideal $\mathcal{I}$ on ω (or its dual filter $\mathcal{I}^*$) is *ω-diagonalizable* if there is a sequence (A_n) of infinite subsets of ω such that

$$\forall A \in \mathcal{I}\ \exists n\ A \cap A_n = \emptyset.$$

If, moreover, for a certain $\mathcal{A} \subseteq [\omega]^\omega$ all A_n's are members of $\mathcal{A}$, then we say $\mathcal{I}$ *is ω-diagonalizable by elements of $\mathcal{A}$*.

Hrušák [4] calls an ideal $\mathcal{I}$ on ω *countably tall* (or *ω-hitting*) if for every sequence (A_n) of infinite subsets of ω there is an $A \in \mathcal{I}$ such that $A_n \cap A$ is infinite for all n. Note that $\mathcal{I}$ is ω-diagonalizable if and only if $\mathcal{I}$ is *not* countably tall.

In this section we characterize selectivity and density of ideals in terms of ω-diagonalizations. We begin with a general observation.

Proposition 3.1. Let $\mathcal{A} \subseteq [\omega]^\omega$ be closed under finite modifications. Then $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{A}$ if and only if there is an $\mathcal{A}$-tree with all branches in $\mathcal{I}^+$.

Proof. First let (A_n) be a sequence of elements of $\mathcal{A}$ such that

$$\forall A \in \mathcal{I}\ \exists n \quad A \cap A_n = \emptyset.$$

Define an $\mathcal{A}$-tree inductively as follows:

- $succ_T(\emptyset) = A_0$,
- at step $n > 0$, having defined $s = \{k_0, \dots, k_{n-1}\} \in T$ so that $k_0 < \dots < k_{n-1}$ and $k_i \in A_i$ for every $i < n$, let

$$succ_T(s) = \{k \in A_n : k > \max(s)\}.$$

Clearly, T is an $\mathcal{A}$-tree. If $A \in [T]$, then since $A \cap A_n \neq \emptyset$ for each $n \in \omega$, $A \in \mathcal{I}^+$.

For the other implication let T be an $\mathcal{A}$-tree with all branches in $\mathcal{I}^+$. For every $s \in T$ let $A_s = succ_T(s)$. Then $\{A_s : s \in T\}$ is a countable family of elements of $\mathcal{A}$ and it suffices to note that every set $A \in [\omega]^\omega$ such that $A \cap A_s \neq \emptyset$ for each $s \in T$ contains a branch of T and so is in $\mathcal{I}^+$. □

In the following sequence of results we deal with ω-diagonalizable ideals imposing increasingly stronger conditions on the diagonalizing sequences. The idea is to localize on the spectrum thus obtained ideals with combinatorial properties studied in this paper: selective properties on one hand and density on the other. Selective properties described in terms of the existence of large branches of certain trees should be compared with Grigorieff's characterization of selective ideals stated in Theorem 1.9.

We are mostly concerned with Borel ideals in which case determinacy arguments can be applied to ideal-related games studied by Laflamme [9]. More precisely, the game $G(\mathcal{X}, \omega, \mathscr{X})$, where in this paper $\mathcal{X} = \mathcal{J}^*$ and $\mathscr{X}$ is either $\mathcal{I}$ or $\mathcal{I}^+$ for certain ideals $\mathcal{J}$ and $\mathcal{I}$ on ω, is played by two players **I** and **II** as follows: at stage $k \in \omega$ player **I** chooses $X_k \in \mathcal{X}$ and then player **II** responds with $n_k \in X_k$. After ω-many steps, **II** wins if $\{n_k : k \in \omega\} \in \mathscr{X}$.

Following Hrušák [4] we say that an ideal $\mathcal{I}$ on D is a *Q-ideal* if for every partition $\{F_n : n < \omega\}$ of D into finite sets there is a selector in $\mathcal{I}^+$. Note that an ideal $\mathcal{I}$ on ω is a Q^+-ideal (see Example 1.6) if and only if $\mathcal{I}|A$ is a Q-ideal for all $A \in \mathcal{I}^+$.

Proposition 3.2. (Meza-Alcántara [14] and [4]) If $\mathcal{I}$ is a Borel ideal on ω, then $\mathcal{I}$ is ω-diagonalizable if and only if $\mathcal{I}$ is a Q-ideal.

Proof. The ideal $\mathcal{I}$ being Borel, the game $G(([\omega]^{<\omega})^*, \omega, \mathcal{I}^+)$ is determined. Now the equivalence is an immediate consequence of the following characterization of Laflamme (cf. [9], Theorem 2.2): **I** has a winning strategy if and only if $\mathcal{I}$ is not a Q-ideal and **II** has a winning strategy if and only if $\mathcal{I}$ is ω-diagonalizable. □

Following Laflamme [9] we say that an ideal $\mathcal{I}$ on ω is *weakly Ramsey* if every $\mathcal{I}^*$-tree has a branch in $\mathcal{I}^+$. By Theorem 1.9, every selective ideal is weakly Ramsey.

Proposition 3.3. If $\mathcal{I}$ is a Borel ideal on ω, then $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{I}^+$ if and only if $\mathcal{I}$ is weakly Ramsey.

Proof. The ideal $\mathcal{I}$ being Borel, the game $G(\mathcal{I}^*, \omega, \mathcal{I}^+)$ is determined. Now the equivalence is an immediate consequence of the following characterization of Laflamme (cf. [9], Theorem 2.7): **I** has a winning strategy if and only if $\mathcal{I}$ is not weakly Ramsey and **II** has a winning strategy if and only if $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{I}^+$. □

Proposition 3.4. For an ideal $\mathcal{I}$ on ω the following are equivalent:
(i) $\mathcal{I}$ is ω-diagonalizable by elements of a filter contained in $\mathcal{I}^+$.
(ii) $\mathcal{I}$ is ω-diagonalizable by elements of a filter on ω.
(iii) The filter $\mathcal{I}^*$ is diagonalizable by a single set $P \in [\omega]^\omega$ in the sense that $P \subseteq^* B$ for every $B \in \mathcal{I}^*$.
(iv) $\mathcal{I}$ is not dense.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Let (A_n) be a sequence of elements of a filter $\mathcal{F}$ such that

$$\forall A \in \mathcal{I}\ \exists n\ A \cap A_n = \emptyset. \tag{1}$$

Let $P \in [\omega]^\omega$ be such that $P \subseteq^* A_n$ for every $n \in \omega$. By (1), the set P diagonalizes the filter $\mathcal{I}^*$.

(iii) $\Rightarrow$ (iv). Let $P \in [\omega]^\omega$ be such that $P \subseteq^* B$ for every $B \in \mathcal{I}^*$. Clearly, P is a witness that $\mathcal{I}$ is not dense.

(iv) $\Rightarrow$ (i). $\mathcal{I}$ being not dense, there is $B \in [\omega]^\omega$ with $[B]^\omega \subseteq \mathcal{I}^+$. Let $A_n = B \setminus n$ for each $n \in \omega$. Then (A_n) diagonalizes $\mathcal{I}$. Indeed, if $A \cap A_n \neq \emptyset$ for every $n \in \omega$, then $A \cap B$ is infinite so $A \in \mathcal{I}^+$. Moreover, the filter generated by the family $\{A_n : n \in \omega\}$ is contained in $\mathcal{I}^+$. □

Following Krawczyk [6] (cf. [21]) we say that an ideal $\mathcal{I}$ (or the associated coideal $\mathcal{I}^+$) is *bisequential* if for every ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$ the ideal $\mathcal{I}$ is ω-diagonalizable by elements of $\mathscr{U}$ (this property is closely related to the notion, introduced by Michael [15] (see also [6] and [16]), of a topological space being bisequential at a point).

The following theorem of Todorčević reveals a connection between selective and bisequential ideals. For the sake of completeness we give its short proof based on results of Section 2.

Theorem 3.5. (Todorčević, [21]) If $\mathcal{I}$ is a selective and analytic (or coanalytic) ideal on ω, then $\mathcal{I}$ is bisequential.

Proof. Fix an arbitrary ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$. By Theorem 2.3, the ideal $\mathcal{I}$ being selective is $\tau_{\mathscr{U}}$-nowhere dense, hence, by 1.11, it is completely $\mathscr{U}$-Ramsey null. In particular, there is a $\mathscr{U}$-tree T with $[T] \subseteq \mathcal{I}^+$. But this, by Proposition 3.1, means that $\mathcal{I}$ is ω-diagonalizable by elements of $\mathscr{U}$. □

Note that the above argument could be easily reversed: if an analytic (or coanalytic) ideal $\mathcal{I}$ is bisequential and $\mathscr{U} \subseteq \mathcal{I}^+$ is an ultrafilter, then (by Proposition 3.1) there is a $\mathscr{U}$-tree T with $[T] \subseteq \mathcal{I}^+$ witnessing that $\mathcal{I}$ is not dense in $\tau_{\mathscr{U}}$. Hence, by Lemma 2.2, $\mathcal{I}$ is nowhere dense in $\tau_{\mathscr{U}}$ which, by Theorem 2.3, means that $\mathcal{I}$ is selective.

The next result proves a stronger implication showing that the definability condition on $\mathcal{I}$ is irrelevant.

Theorem 3.6. If $\mathcal{I}$ is a bisequential ideal on ω, then $\mathcal{I}$ is selective.

Proof. Let $\{X_n : n \in \omega\}$ be a partition of ω no finite union of elements of which is in the filter $\mathcal{I}^*$. For each $k \in \omega$ let $Y_k = \bigcup_{n>k} X_n$. Since (Y_k) is a decreasing sequence of elements of the coideal I^+, there is an ultrafilter $\mathscr{U} \subseteq \mathcal{I}^+$ such that $Y_k \in \mathscr{U}$ for every $k \in \omega$.

Using the fact that $\mathcal{I}$ is bisequential fix a sequence (A_n) of elements of $\mathscr{U}$ such that

$$\forall A \in \mathcal{I}\ \exists n \quad A \cap A_n = \emptyset. \tag{1}$$

We will find a desirable selector of $\{X_n : n \in \omega\}$ inductively picking some of its elements s_n, $n \in \omega$, in such a way that:

- $\forall n \in \omega\ s_n \in A_n$,
- $\forall i < n \forall m\ (s_i \in X_m \Rightarrow s_n \notin X_m)$.

More precisely, at step n let $m_n = \min\{k :\ \forall i < n\ s_i \notin Y_k\}$. Since the set $A_n \cap Y_{m_n}$ is non-empty as a member of $\mathscr{U}$, choose s_n as one of its elements.

Finally, if $S = \{s_n : n \in \omega\}$, then $|S \cap X_n| \leq 1$ and $S \cap A_n \neq \emptyset$ for each $n \in \omega$. By (1), the latter proves that $S \in \mathcal{I}^+$. □

Corollary 3.7. If $\mathcal{I}$ is an analytic (or coanalytic) ideal on ω, then $\mathcal{I}$ is bisequential if and only if $\mathcal{I}$ is selective.

Corollary 3.8. If $\mathcal{I}$ is a bisequential ideal on ω, then $\mathcal{I}$ is not dense. In particular, if $\mathcal{I}$ is selective and analytic (or coanalytic), then $\mathcal{I}$ is not dense.

Proof. The first part immediately follows from Proposition 3.4. The second one is implied by the first with the help of Corollary 3.7. □

We end this section with a comment on the strongest of diagonalizing principles under consideration.

Proposition 3.9.

(i) If $\mathcal{I}$ is an ideal on ω, then $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{I}^*$ if and only if $\mathcal{I}$ is countably generated.
(ii) If, moreover, $\mathcal{I}$ is Borel, then $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{I}^*$ if and only if every $\mathcal{I}^+$-tree has a branch in $\mathcal{I}^+$.

Proof. Part (i) is obvious. To prove part (ii) note that the ideal $\mathcal{I}$ being Borel, the game $G(\mathcal{I}^*, \omega, \mathcal{I})$ is determined. Now the equivalence is an immediate consequence of the following characterization of Laflamme (cf. [9], Theorem 2.9): **I** has a winning strategy if and only if $\mathcal{I}$ is countably generated and **II** has a winning strategy if and only if there is an $\mathcal{I}^+$-tree with all branches in $\mathcal{I}$. □

The following table summarizes the results of this section presenting increasingly stronger diagonalizability principles and the corresponding properties of Borel ideals on ω.

$\mathcal{I}$ is ω-diagonalizable by elements of	$\mathcal{I}$ is
$[\omega]^\omega$	a Q-ideal
$\mathcal{I}^+$	weakly Ramsey
a filter on ω	not dense
$\forall \mathscr{U} \subseteq \mathcal{I}^+$ the ultrafilter $\mathscr{U}$	selective
the ultrafilter $\mathcal{I}^*$	countably generated

Remark 3.10. Using unfolded versions of the appropriate Laflamme games we can generalize Propositions 3.2 and 3.3 to coanalytic ideals and Proposition 3.9(ii) to analytic ideals. Indeed, consider the game $G(\mathcal{I}^*, \omega, \mathcal{Z})$, where $\mathcal{I}$ is an ideal on ω and $\mathcal{Z} \subset [\omega]^\omega$ is analytic. There is a closed set $\mathcal{Z}' \subset [\omega]^\omega \times \omega^\omega$ such that $\mathcal{Z}$ is the projection of $\mathcal{Z}'$. The unfolded game $G'(\mathcal{I}^*, \omega, \mathcal{Z}')$ is played by two players I and II as follows: at stage $k \in \omega$ player I chooses $X_k \in \mathcal{I}^*$ and then player II responds with a pair (n_k, m_k) with $n_k \in X_k$ and $m_k \in n \cup \{-1\}$. After ω-many steps II wins if $(\{n_k : k \in \omega\}, \bar{m}) \in \mathcal{Z}'$, where $\bar{m}$ is the sequence of those m_k which are not equal to -1. Since $\mathcal{Z}'$ is closed, the game $G'(\mathcal{I}^*, \omega, \mathcal{Z}')$ is determined. Moreover, one can prove that if player I (respectively: II) has a winning strategy in $G'(\mathcal{I}^*, \omega, \mathcal{Z}')$, then he also has a winning strategy in $G(\mathcal{I}^*, \omega, \mathcal{Z})$ (cf. the proof of [8, Theorem 1.6]). Hence, the game $G(\mathcal{I}^*, \omega, \mathcal{Z})$ is determined for all analytic sets $\mathcal{Z}$.

4. Weak selectivity via ω-diagonalization

Recall that an ideal $\mathcal{I}$ on D is

- weakly selective if for every set $X \in \mathcal{I}^+$ and every partition $\{F_n : n < \omega\}$ of X into sets from $\mathcal{I}$ there is a selector in $\mathcal{I}^+$ (see Example 1.7). Equivalently, for every partition $\{A_n : n \in \omega\}$ of D no finite union of elements of which is in the dual filter of $\mathcal{I}$

and at most one element of the partition is not in $\mathcal{I}$ there is a selector in $\mathcal{I}^+$,

- weakly Ramsey if every $\mathcal{I}^*$-tree has a branch in $\mathcal{I}^+$.

A connection between these two properties is revealed by the following characterization.

Theorem 4.1. (Grigorieff [3]) An ideal $\mathcal{I}$ on ω is weakly selective if and only if the ideal $\mathcal{I}|A$ is weakly Ramsey for every $A \in \mathcal{I}^+$.

The next two results are attempts to characterize weak selectivity in terms of ω-diagonalizations.

Proposition 4.2. If $\mathcal{I}$ is a coanalytic ideal on ω, then $\mathcal{I}$ is weakly selective if and only if for each set $A \in \mathcal{I}^+$ the ideal $\mathcal{I}|A$ is ω-diagonalizable by elements of $(\mathcal{I}|A)^+$.

Proof. By the Grigorieff's characterization of weak selectivity (see Theorem 4.1) it suffices to show that for each set $A \in \mathcal{I}^+$ the ideal $\mathcal{I}|A$ is weakly Ramsey if and only if it is ω-diagonalizable by elements of $(\mathcal{I}|A)^+$. This, however, $\mathcal{I}$ being coanalytic, is an immediate consequence of Proposition 3.3 and Remark 3.10. □

Kwela and Sabok [8] (inspired by Todorčević's notion of countably separated gaps; cf. [19]) call an ideal $\mathcal{I}$ on ω *countably separated* if there is a sequence (A_n) of infinite subsets of ω such that

$$\forall A \in \mathcal{I} \forall B \in \mathcal{I}^+ \exists n \quad (A \cap A_n = \emptyset \wedge B \cap A_n \in \mathcal{I}^+).$$

(Hence, in particular, (A_n) witnesses that $\mathcal{I}$ is ω-diagonalizable.)

Kwela and Sabok proved (see [8, Theorem 1.1]) that countable separability together with density characterize, up to an isomorphism, ideals which are topologically represented (in the sense of Sabok and Zapletal – see Example 1.7).

Proposition 4.3. Every countably separated ideal on ω is weakly selective.

Proof. Assume that a sequence (A_n) witnesses the countable separability of an ideal $\mathcal{I}$. Then, for each set $A \in \mathcal{I}^+$, an obvious modification of this sequence witnesses that the ideal $\mathcal{I}|A$ is ω-diagonalizable by elements of $(\mathcal{I}|A)^+$. By Theorem 2.7 of [9], this in turn implies, that $\mathcal{I}|A$ is weakly Ramsey (note that this implication does not require the assumption that $\mathcal{I}$ is Borel – see the proof of Proposition 3.3). Consequently, by Theorem 4.1, $\mathcal{I}$ is weakly selective. □

In view of Proposition 4.2 it is tempting to conjecture that, at least for a Borel ideal $\mathcal{I}$ on ω, the condition that the ideal $\mathcal{I}$ is ω-diagonalizable by elements of $\mathcal{I}^+$ characterizes ideals with the following selective property: for every partition of ω into sets in $\mathcal{I}$ there is a selector in $\mathcal{I}^+$. This is, however, not the case: by results of Kwela

[7], being weakly Ramsey is a provably stronger condition than the selective property, formulated above.

Let us note that the implication from Proposition 4.3 cannot be reversed.

Example 4.4. Let $\mathcal{I}$ be either the ideal on ω generated by a MAD family (see Example 1.2) or a maximal selective ideal on ω (see Example 1.4). Then $\mathcal{I}$ is dense and selective, hence weakly selective but not countably separated. Otherwise, by the theorem of Kwela and Sabok mentioned above (see [8, Theorem 1.1]), the ideal $\mathcal{I}$ would be isomorphic to a topologically represented ideal which, however, would contradict its selectivity (see Example 1.7).

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