HAAR-SMALLEST SETS

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Abstract. Recently, Banakh et al. unified the notions of Haar-null and Haar-meager sets: for a collection $\mathcal{I}$ of subsets of 2^ω, we say that a subset A of an abelian Polish group X is Haar-$\mathcal{I}$ ($A \in \mathcal{HI}$) if there are a Borel hull $B \supseteq A$ and a continuous map $f\colon 2^\omega \to X$ with $f^{-1}[B+x] \in \mathcal{I}$ for all $x \in X$. It turns out that for $\mathcal{I} = \mathcal{N}$ we obtain Haar-null sets. The same holds for $\mathcal{I} = \mathcal{M}$ and Haar-meager sets.

As $\mathcal{I} \subseteq \mathcal{J}$ implies $\mathcal{HI} \subseteq \mathcal{HJ}$, the above notion enables to develop a whole hierarchy of Haar-null and Haar-meager sets. This requires constructing examples distinguishing $\mathcal{HI}$ for various $\mathcal{I}$.

In this paper we are interested in several particular collections $\mathcal{I}$: the σ-ideal of countable sets, the ideal of finite sets and the collections of sets of cardinality at most n. We study basic properties of the corresponding families $\mathcal{HI}$, give suitable examples distinguishing them (in all abelian Polish groups of the form $\mathbb{R} \times X$ or $[0,1] \times X$ as well as in $C(K)$, where K is compact metrizable, and ℓ_∞) and study σ-ideals generated by compact Haar-$\mathcal{I}$ sets for the considered collections $\mathcal{I}$. Moreover, we answer some questions asked in [2] by Banakh and Jabłońska concerning null-finite sets and pose several open problems.

1. Introduction

By Haar's Theorem, in a locally compact topological group X there is, up to a positive multiplicative constant, a unique countably additive nontrivial measure μ on the Borel subsets of X, satisfying some regularity properties and such that $\mu(x+B) = \mu(B)$ for each $x \in X$ and Borel $B \subseteq X$ (see [10]). This measure is called left Haar measure, or simply Haar measure if X is abelian. In the case of the real line or the Cantor cube 2^ω it is equal to the standard Lebesgue measure. In the absence of Haar measure (or any other reasonable translation-invariant measure) in Polish groups which are not locally compact, a good approach is to concentrate only on the family of subsets which are supposed to be null (without indicating a measure). In [5] Christensen introduced the notion of Haar-null sets – he called a subset A of an abelian Polish group X *Haar-null* provided that there are a Borel set $B \supseteq A$ and a Borel σ-additive probability measure μ on X such that $\mu(B+x) = 0$ for all $x \in X$. A big advantage of this concept is that in a locally compact group it is equivalent to the notion of Haar measure zero sets and at the same time it can be used in a significantly larger class of groups. In [9] Hunt, Sauer and Yorke, unaware of Christensen's paper, reintroduced the notion of Haar-null sets in the context of dynamical systems (in their paper Haar-null sets are called shy sets and their complements are called prevalent sets). This shows that Haar-null sets are very natural and useful. Since that time this notion has been applied in diverse fields of mathematics including analysis, dynamical systems, functional analysis, group theory and set theory.

A topological analogue of Haar-null sets was defined by Darji in [6] – a subset A of an abelian Polish group X is *Haar-meager* provided that there are a Borel set $B \supseteq A$, a compact metric space K and a continuous map $f\colon K \to X$ with $f^{-1}[B+x]$ meager in K for each $x \in X$. Similarly as before, in the locally compact case this notion is equivalent to the notion of meager sets. However, Haar-meager sets enable us to capture in a single notion of smallness both group and topological structures.

Recall that a collection of subsets of a set X is called:

- a *semi-ideal* on X, if it is closed under taking subsets;
- an *ideal* on X, if it is closed under taking subsets and finite unions;
- a *σ-ideal* on X, if it is closed under taking subsets and countable unions.

It is known that Haar-null and Haar-meager sets form σ-ideals (cf. [5], [6] and [9]).

Recently, the notions of Haar-null sets and Haar-meager sets were unified by Banakh et al. in [3] by introducing the concept of Haar-small sets. For a semi-ideal $\mathcal{I}$ on the Cantor cube 2^ω, we say that a subset A of an abelian Polish group X is *Haar-$\mathcal{I}$* ($A \in \mathcal{HI}$) if there are a Borel hull $B \supseteq A$ and a continuous map $f\colon 2^\omega \to X$ with $f^{-1}[B+x] \in \mathcal{I}$ for all $x \in X$. Obviously, the collection of Haar-$\mathcal{I}$ subsets of an abelian Polish group is a semi-ideal. It turns out that if $\mathcal{I}$ is the σ-ideal $\mathcal{N}$ of subsets of 2^ω of Lebesgue's measure zero, then we obtain Haar-null sets (cf. [3]; to shed some light on this fact let us point out that by [1] for any strictly positive continuous measures μ and ν on the Cantor cube 2^ω the corresponding σ-ideals of null sets are topologically isomorphic). The same holds for the σ-ideal $\mathcal{M}$ of meager subsets of 2^ω and Haar-meager sets (cf. [3]).

In this paper we will restrict ourselves only to the following semi-ideals:

- the σ-ideal of countable subsets of 2^ω (we call the corresponding Haar-small sets *Haar-countable* and denote their collection by $\mathcal{H}\mathrm{Ctbl}$);
- the ideal of finite subsets of 2^ω (we call the corresponding Haar-small sets *Haar-finite* and denote their collection by $\mathcal{H}\mathrm{Fin}$);
- the semi-ideals of subsets of 2^ω of cardinality at most n, for $n \in \omega$ (we call the corresponding Haar-small sets *Haar-n* and denote their collection by $\mathcal{H}n$).

Clearly,

$$\text{Haar-}n \implies \text{Haar-}(n+1) \implies \text{Haar-finite} \implies \text{Haar-countable}$$

for any $n \in \omega$. Moreover, each Haar-countable set is Haar-null and Haar-meager. As a consequence, a countable union of Haar-countable sets cannot be the whole space.

Hunt, Sauer and Yorke introduced Haar-null sets in order to state mathematically precisely sentences saying that some property holds for almost all elements of a given infinite-dimensional linear space such as $C([0,1])$ or $L^1([0,1])$. This was supposed to be a counterpart of sets of measure zero in such spaces. However, Haar-null sets and Haar-meager sets allow us only to say that some property is rare and we cannot put it on a scale and compare to other rare properties. In other words, we do not know how to say that some property is "more rare" than another one. Although, this can be obtained using Haar-small sets, as $\mathcal{I} \subseteq \mathcal{J}$ implies $\mathcal{HI} \subseteq \mathcal{HJ}$ – we can look for a semi-ideal $\mathcal{I}$ contained in $\mathcal{N}$ such that the set of points satisfying one property is Haar-$\mathcal{I}$ and the set of points satisfying the second one is not. Thus, the notion of Haar-small sets enables us to develop a whole hierarchy of Haar-null or Haar-meager subsets of abelian Polish groups. Next two Propositions illustrate the above by distinguishing two of the rare properties which motivated Hunt, Sauer and Yorke for introducing Haar-null sets. Note that the set $\mathcal{A}$ of functions with $\int_0^1 f(x)dx = 0$ is Haar-null in $L^1[0,1]$ by [9] and the set $\mathcal{SD}$ of somewhere differentiable functions (i.e., functions which have a derivative at some point) is Haar-null in $C[0,1]$ by [8].

Proposition 1. *The set $\mathcal{A}$ of functions with $\int_0^1 f(x)dx = 0$ is Haar-1 in $L^1[0,1]$ (as well as in $C[0,1]$).*

Proof. Let $C \subseteq \mathbb{R}$ denote the standard ternary Cantor set. For each $c \in C$ define $\phi(c)\colon [0,1] \to \mathbb{R}$ by $\phi(c)(x) = c$ for all $x \in [0,1]$. It is easy to check that $\phi\colon C \to L^1[0,1]$ is a continuous map.

As C and 2^ω are homeomorphic, it suffices to show that for every $h \in L^1[0,1]$ the set $\phi^{-1}[\mathcal{A} + h]$ has cardinality 1 or, in other words, for every $h \in L^1[0,1]$ we have $\phi(c) - h \in \mathcal{A}$ for at most one $c \in C$. Fix any $h \in L^1[0,1]$ and suppose that $\int_0^1 (\phi(c) - h)(x)dx = 0$ for some $c \in C$. Then for each $c' \in C$, $c' \neq c$, we have $\int_0^1 (\phi(c') - h)(x)dx = c' - c \neq 0$. Hence, ϕ is the desired map.

As all continuous functions are integrable, in the case of $C[0,1]$ the proof is exactly the same. ☐

Proposition 2. *The set $\mathcal{SD}$ of somewhere differentiable functions is not Haar-finite in $C[0,1]$.*

Proof. Let $\phi\colon 2^\omega \to C[0,1]$ be a continuous map. We need to show that $\phi^{-1}[\mathcal{SD}+h]$ is infinite for some $h \in C[0,1]$.

If $\phi[2^\omega]$ is finite, then let $h \in C[0,1]$ be such that $\phi^{-1}[\{h\}]$ is infinite and observe that $h \in \mathcal{SD} + h$. Hence, $\phi^{-1}[\mathcal{SD} + h]$ is infinite as well.

Assume now that $\phi[2^\omega]$ is infinite and take any injective convergent sequence $(f_n) \in \phi[2^\omega]$. Denote $f = \lim_n f_n$. Fix a sequence of closed intervals (I_n) such that $I_n \subseteq (\frac{1}{n+2}, \frac{1}{n+1})$ for each $n \in \omega$. Let $J_n = [\max I_{n+1}, \min I_n]$ and $g_n\colon J_n \to \mathbb{R}$ be the linear function given by $g_n(\max I_{n+1}) = 0$ and $g_n(\min I_n) = (f_n - f_{n+1})(\min I_n)$. Define $h\colon [0,1] \to \mathbb{R}$ by:

- $h \restriction [\max I_0, 1] = f_0(\max I_0)$;
- $h \restriction I_n = f_n \restriction I_n$ for each $n \in \omega$;
- $h \restriction J_n = (f_{n+1} + g_n) \restriction J_n$ for each $n \in \omega$;
- $h(0) = f(0)$.

Note that the function h is continuous (continuity in 0 follows from $\lim_n f_n = f$ and $\lim_n \sup_{x \in J_n} g_n(x) = 0$) and $f_n \in \mathcal{SD} + h$ for each $n \in \omega$ (as $(f_n - h) \restriction I_n$ is constant). Since (f_n) is injective, we conclude that $\phi^{-1}[\mathcal{SD} + h]$ is infinite. ☐

Actually, in the forthcoming paper [11] we use Haar-smallest sets in further studies of differentiability of continuous functions. Namely, we prove that $\mathcal{SD}$ is even not Haar-countable, the set of continuous functions that are differentiable on a set of positive measure is Haar-countable but not Haar-finite, and the set of continuous functions which are differentiable on a set of full measure is Haar-1.

Although the concept of Haar-small sets is very recent, its variation described below had already been applied to solve an old problem of Baron and Ger from [4]. This shows the potential impact of Haar-small sets. We say that a subset A of an abelian Polish group X is *null-finite* if there is a convergent sequence $(x_n)_{n\in\omega} \subseteq X$

such that $\{n \in \omega : \ x_n - x \in A\}$ is finite for all $x \in X$. This notion is very closely related to Haar-finite sets: in Section 2 we characterize Haar-finite Borel sets $B \subseteq X$ as those for which there is a homeomorphic copy $D \subseteq X$ of the Cantor space 2^ω with $(D-x)\cap B$ finite for each $x \in X$. Therefore, a witness for a null-finite set is compact, but is does not have to be uncountable as in the case of Haar-finite sets. Clearly, each Haar-finite set is null-finite.

Null-finite sets were introduced by Banakh and Jabłońska in order to solve the mentioned problem of Baron and Ger concerning functional equations: a mid-point convex function $f\colon G \to \mathbb{R}$ defined on an open convex subset G of a complete linear metric space X is continuous if it is upper bounded on a Borel subset $B \subseteq G$ which is neither Haar-null nor Haar-meager in X.

The main aim of this paper is to develop a hierarchy of Haar-null sets. This requires constructing examples distinguishing $\mathcal{HI}$ for various semi-ideals $\mathcal{I}$. This paper studies relationships between families $\mathcal{H}n$, $\mathcal{H}\mathrm{Fin}$, $\mathcal{H}\mathrm{Ctbl}$ and $\mathcal{HN} \cap \mathcal{HM}$. We give such examples for the considered semi-ideals on the real line. However, in Section 2, besides establishing some useful characterizations, we show that our results can be applied to a wider class of abelian Polish groups including $C(K)$ (where K is compact metrizable), ℓ^∞ and the ones of the form $\mathbb{R} \times X$ or $[0,1] \times X$.

In Section 3 we show that all countable sets are Haar-1, characterize closed Haar-1 subsets of $\mathbb{R}$ and construct an uncountable compact Haar-1 set. In Section 4 we distinguish $\mathcal{H}2$ from $\mathcal{H}1$ by showing that the standard ternary Cantor set C is Haar-2 but not Haar-1. Also, we conclude that a compact Haar-2 set does not have to be a countable union of compact Haar-1 sets. Section 5 is devoted to proving that the sequence $\mathcal{H}1 \subseteq \mathcal{H}2 \subseteq \ldots$ does not stabilize.

In the longest and most technical Section 6, answering a question posed by Swaczyna during his talk on XLI Summer Symposium in Real Analysis (Wooster, 2017), we show that the family of all Haar-finite sets is not an ideal. This means that $\mathcal{I}$ being an ideal does not determine whether $\mathcal{HI}$ is an ideal. What is more, we get an example of a compact Haar-finite set which is not Haar-n for any n.

In the last Section we construct a compact Haar-countable set which is not Haar-finite and a compact null and meager set which is neither Haar-countable nor a countable union of compact Haar-countable sets. Thus, the σ-ideal generated by compact Haar-countable sets is a proper subset of $\mathcal{N} \cap \mathcal{M}$ (which is the same as $\mathcal{HN} \cap \mathcal{HM}$, since we are on $\mathbb{R}$).

All our results concerning Haar-finite sets can be adapted also for null-finite sets. Consequently, we show that the family of all null-finite sets is not an ideal. This answers a question posed in the first version of [2] and asked by Banakh during his talk at the conference Frontiers of Selection Principles (Warsaw, 2017). Moreover, in Section 7 we construct a compact null and meager set which is not a countable union of compact null-finite sets. This is a partial solution to another problem from [2]. Actually, we do not know whether there is a Borel null-finite subset of $\mathbb{R}$ which is not Haar-finite.

2. Preliminaries

We follow standard set-theoretic and topological notation and terminology (see [10]). In particular, by ω we denote the set $\{0,1,\ldots\}$, identify each $n \in \omega$ with the set $\{0,1,\ldots,n-1\}$ and denote by $|X|$ the cardinality of a set X.

If $\mathcal{A} \subseteq \mathcal{P}(X)$, then $\sigma\mathcal{A}$ is the σ-ideal generated by $\mathcal{A}$, i.e., the family of sets which can be covered by a countable union of elements of $\mathcal{A}$. Moreover, if X is a topological space, then $\overline{\mathcal{A}}$ denotes the family of compact sets belonging to $\mathcal{A}$.

Note that $\mathcal{H}\mathrm{Ctbl} = \mathcal{H}[2^\omega]^{\leq\omega}$, $\mathcal{H}\mathrm{Fin} = \mathcal{H}[2^\omega]^{<\omega}$ and $\mathcal{H}n = \mathcal{H}[2^\omega]^{\leq n}$. In our further considerations we will use the following equivalence. By a Cantor set we mean a homeomorphic copy of the space 2^ω.

Proposition 3. *Let α stand for one of the symbols $\leq \omega$, $< \omega$ or $\leq n$ for some $n \in \omega$. Then the following are equivalent for any Borel subset B of an abelian Polish group X:*

(a) *$B \in \mathcal{H}[2^\omega]^\alpha$;*
(b) *there is a Cantor set $D \subseteq X$ with $(D - x) \cap B \in [X]^\alpha$ for each $x \in X$;*
(c) *there is a Cantor set $D \subseteq X$ such that for any $S \subseteq D$ with $S \notin [X]^\alpha$ we have $\bigcap_{s\in S}(B - s) = \emptyset$.*

Proof. Firstly, we show that (a) implies (b). Let $f\colon 2^\omega \to X$ be continuous and such that $f^{-1}[B + x] \in [2^\omega]^\alpha$ for all $x \in X$. Without loss of generality we can assume that $f[2^\omega]$ is uncountable (otherwise f could be a witness only for $B = \emptyset$ and this case is easy). As $f[2^\omega]$ is also compact, we can find a Cantor set $D \subseteq f[2^\omega]$ (see [10]). Now it suffices to observe that

$$|f^{-1}[B + x]| \geq |f[2^\omega] \cap (B + x)| \geq |D \cap (B + x)| = |(D - x) \cap B|$$

for every $x \in X$, so $f^{-1}[B + x] \in [2^\omega]^\alpha$ implies $(D - x) \cap B \in [X]^\alpha$.

Now we show that (b) implies (a). Let $D \subseteq X$ be a Cantor set with $(D-x)\cap B \in X^\alpha$ for each $x \in X$. Take a homeomorphism $f\colon 2^\omega \to D$. We have

$$|(D - x) \cap B| = |f[2^\omega] \cap (B + x)| = |f^{-1}[B + x]|$$

for every $x \in X$ (the last equality is due to injectivity of f), so $(D - x) \cap B \in [X]^\alpha$ implies $f^{-1}[B + x] \in [2^\omega]^\alpha$.

Finally, we deal with equivalence of (b) and (c). It suffices to show that for any $D \subseteq X$ the condition $(D - x) \cap B \in [X]^\alpha$ for each $x \in X$ is equivalent to $\bigcap_{s\in S}(B - s) = \emptyset$ for each $S \subseteq D$ with $S \notin [X]^\alpha$. Indeed, both assertions can be written as: for every $x \in X$ and every $S \subseteq D$ with $S \notin [X]^\alpha$ there is such $s \in S$ that $s - x \notin B$. $\square$

We will often use item (c) of the above characterization as a definition of $\mathcal{H}[2^\omega]^\alpha$.

Remark 1. Observe that $\mathcal{H}[2^\omega]^{\leq\omega} = \mathcal{H}[2^\omega]^{<\mathfrak{c}}$. Indeed, a set A is in $\mathcal{H}[2^\omega]^{<\mathfrak{c}}$ provided that there are a Borel $B \supseteq A$ and a continuous map $\phi\colon 2^\omega \to X$ with $|\phi[2^\omega] \cap (B + x)| < \mathfrak{c}$ for each $x \in X$. However, in this case the set $\phi[2^\omega] \cap (B + x)$ is countable as a Borel set (cf. [10]).

Remark 2. A linear copy of a Haar-$\mathcal{I}$ subset of $\mathbb{R}$ is still Haar-$\mathcal{I}$. Indeed, if $X \in \mathcal{H}\mathcal{I}$ is witnessed by $\phi\colon 2^\omega \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = ax + b$, for some $a, b \in \mathbb{R}$, then define $\psi\colon 2^\omega \to \mathbb{R}$ by $\psi(\alpha) = f(\phi(\alpha))$, for all $\alpha \in 2^\omega$, and observe that $\psi^{-1}[f[X] + r] = \phi^{-1}[X + \frac{r}{a}] \in \mathcal{I}$ (in the case $a = 0$ the set $f[A]$ also is in $\mathcal{H}\mathcal{I}$ since it has only one element). We will use this observation several times.

The following series of results shows that all results of this paper can be applied to $C(K)$ (where K is a compact metrizable space), ℓ^∞ and each abelian Polish group of the form $\mathbb{R} \times X$ or $[0, 1] \times X$ (in the consecutive Sections we deal only with subsets of $[0, 1]$). In particular, an example for $\mathcal{H}\mathcal{I} \neq \mathcal{H}\mathcal{J}$ on $[0, 1]$ establishes also an example for $\mathcal{HI} \neq \mathcal{HJ}$ in each of those spaces. It should be mentioned that most of our reasonings work also in some countable products of finite groups.

Proposition 4. *Let X and Y be two nonempty abelian Polish groups. Fix a semi-ideal $\mathcal{I}$ on 2^ω and $A \subseteq X$.*

(i) *If $A \in \mathcal{HI}$ ($\overline{\mathcal{HI}}$), then $A \times Y \in \mathcal{HI}$ ($\overline{\mathcal{HI}}$).*
(ii) *If $A \notin \mathcal{HI}$ ($\overline{\mathcal{HI}}$), then $A \times Y \notin \mathcal{HI}$ ($\overline{\mathcal{HI}}$).*
(iii) *If $A \in \sigma\overline{\mathcal{HI}}$, then $A \times Y \in \sigma\overline{\mathcal{HI}}$.*

Proof. Note that A is closed in X if and only if $A \times Y$ is closed in $X \times Y$. Therefore, we only need to prove the assertions for $\mathcal{HI}$.

(i): Let $\phi\colon 2^\omega \to X$ be a witness for $A \in \mathcal{HI}$. Fix any $y \in Y$ and define $\psi\colon 2^\omega \to X \times Y$ by $\psi(\alpha) = (\phi(\alpha), y)$ for each $\alpha \in 2^\omega$. Then ψ is continuous and for each $(r, r') \in X \times Y$ we have $\psi^{-1}[A \times Y + (r, r')] = \phi^{-1}[A + r] \in \mathcal{I}$.

(ii): Let $\psi\colon 2^\omega \to X \times Y$ be continuous. Then $\phi\colon 2^\omega \to X$ given by $\phi(\alpha) = \pi_X(\psi(\alpha))$, where $\pi_X\colon X \times Y \to X$ is the projection map, is continuous as well. Hence, there is $r \in X$ with $\phi^{-1}[A + r] \notin \mathcal{I}$. Then we have $\psi^{-1}[A \times Y + (r, r')] \supseteq \phi^{-1}[A + r] \notin \mathcal{I}$ for each $r' \in Y$.

(iii): This is an immediate consequence of item (i). ☐

Proposition 5. *Fix a semi-ideal $\mathcal{I}$ on 2^ω and $A \subseteq \mathbb{R}$. Let $A' = \{f \in C(K) : f(0) \in A\}$ (where K is a fixed compact metrizable space) and $A'' = \{(x_n)_n \in \ell^\infty : x_0 \in A\}$.*

(i) *If $A \in \mathcal{HI}$ ($\overline{\mathcal{HI}}$), then $A', A'' \in \mathcal{HI}$ ($\overline{\mathcal{HI}}$).*
(ii) *If $A \notin \mathcal{HI}$ ($\overline{\mathcal{HI}}$), then $A', A'' \notin \mathcal{HI}$ ($\overline{\mathcal{HI}}$).*
(iii) *If $A \in \sigma\overline{\mathcal{HI}}$, then $A', A'' \in \sigma\overline{\mathcal{HI}}$.*

Proof. We will deal with $C(K)$. In the case of ℓ^∞ the proof is almost the same.

Note that A is closed in $\mathbb{R}$ if and only if A' is closed in $C(K)$. Therefore, we only need to prove the assertions for $\mathcal{HI}$.

(i): Let $\phi\colon 2^\omega \to \mathbb{R}$ be a witness for $A \in \mathcal{HI}$. Define $\psi\colon 2^\omega \to C(K)$ by letting $\psi(\alpha)$ be the function constantly equal to $\phi(\alpha)$, for each $\alpha \in 2^\omega$. Then ψ is continuous and for each $g \in C(K)$ we have $\psi^{-1}[A' + g] = \phi^{-1}[A + g(0)] \in \mathcal{I}$.

(ii): Suppose now that $\psi\colon 2^\omega \to C(K)$ is continuous and let $\phi\colon 2^\omega \to \mathbb{R}$ be given by $\psi(\alpha) = \psi(\alpha)(0)$ for each $\alpha \in 2^\omega$. Notice that ϕ is continuous. Thus, there is $r \in \mathbb{R}$ with $\phi^{-1}[A + r] \notin \mathcal{I}$. Let $g\colon K \to \mathbb{R}$ be given by $g(x) = r$ for each $x \in K$. Then $\psi^{-1}[A' + g] = \phi^{-1}[A + r] \notin \mathcal{I}$ and we are done.

(iii): This is an immediate consequence of item (i). ☐

Proposition 6. *Fix two nonempty abelian Polish groups X and Y. Let $\mathcal{I}$ stand for $[2^\omega]^{\leq\omega}$, $[2^\omega]^{<\omega}$ or $[2^\omega]^{\leq n}$ for some $n \in \omega$. Suppose that $A \subseteq X$ is closed and such that $A \cap U \notin \mathcal{HI}$ for each open $U \subseteq X$ with $A \cap U \neq \emptyset$. Then $A \times Y \notin \sigma\overline{\mathcal{HI}}$.*

Proof. Suppose to the contrary that $A \times Y = \bigcup_{n\in\omega} Z_n$ for some sequence of closed Haar-$\mathcal{I}$ sets $(Z_n)_n$. As $A \times Y$ is closed, by the Baire's category theorem, we can find $m \in \omega$ such that Z_m has nonempty interior in the space $A \times Y$ (with the subspace topology). Hence, there are open sets $U \subseteq X$ and $V \subseteq Y$ with $\emptyset \neq (A \cap U) \times V \subseteq Z_m$. We will show that $Z_m \notin \mathcal{HI}$.

Let $\psi\colon 2^\omega \to X \times Y$ be continuous. Consider the continuous function $\psi_Y\colon 2^\omega \to Y$ given by $\psi_Y(\alpha) = \pi_Y(\psi(\alpha))$, where π_Y is the projection map. Find any $y \in Y$ with $\psi_Y^{-1}[V + y] \neq \emptyset$. Then $\psi_Y^{-1}[V + y]$, as an open set, must contain a clopen set $B \subseteq 2^\omega$.

Note that B is homeomorphic to 2^ω and denote this homeomorphism by $\sigma\colon 2^\omega \to B$. Consider $\phi\colon 2^\omega \to X$ given by $\phi(\alpha) = \pi_X(\psi(\sigma(\alpha)))$, where π_X is the projection map. Note that ϕ is continuous and $A \cap U \neq \emptyset$ (otherwise $(A\cap U)\times V$ would be empty). Thus, there is $x \in X$ with $\phi^{-1}[(A\cap U)+x] \notin \mathcal{I}$. Then we have $\psi^{-1}[Z_m + (x,y)] \notin \mathcal{I}$ as

$$|\psi^{-1}[(A\cap U)\times V + (x,y)]| \geq |B \cap \psi^{-1}[\pi_X^{-1}[(A\cap U)+x]]| = |\phi^{-1}[(A\cap U)+x]|.$$

□

Remark 3. Note that for a closed set $A \subseteq \mathbb{R}$ the assumption $A \cap I \notin \mathcal{HI}$, for each open interval $I \subseteq \mathbb{R}$ with $A \cap I \neq \emptyset$, implies $A \notin \sigma\overline{\mathcal{HI}}$. This is a simple consequence of the Baire's category theorem (actually, it also follows from the previous Proposition – it suffices to take any Y of cardinality 1 and observe that $\mathbb{R}\times Y$ is actually the same as $\mathbb{R}$). We will often refer to this basic observation in our further considerations.

Proposition 7. *Let $\mathcal{I}$ stand for $[2^\omega]^{\leq\omega}$, $[2^\omega]^{<\omega}$ or $[2^\omega]^{\leq n}$, for some $n \in \omega$. Suppose that $A \subseteq \mathbb{R}$ is closed and such that $A \cap U \notin \mathcal{HI}$ for each open $U \subseteq \mathbb{R}$ with $A\cap U \neq \emptyset$. Then $A' = \{f \in C(K):\ f(0) \in A\} \notin \sigma\overline{\mathcal{HI}}$ (where K is a fixed compact metrizable space) and $A'' = \{(x_n)_n \in \ell^\infty:\ x_0 \in A\} \notin \sigma\overline{\mathcal{HI}}$.*

Proof. Suppose to the contrary that $A' = \bigcup_{n\in\omega} Z_n$ for some sequence of closed Haar-$\mathcal{I}$ sets $(Z_n)_n$. As A' is closed, by the Baire's category theorem, we can find $m \in \omega$ such that Z_m has nonempty interior in the space A', i.e., there are $h \in C(K)$ and $\varepsilon > 0$ with $\emptyset \neq B(h,\varepsilon)\cap A' \subseteq Z_m$. We will show that $Z_m \notin \mathcal{HI}$.

Find any $x \in A \cap (h(0)-\varepsilon, h(0)+\varepsilon)$ (such x exists since otherwise $B(h,\varepsilon)\cap A'$ would be empty). Without loss of generality we may assume that $x \geq h(0)$. Define $\delta = (h(0)+\varepsilon - x)/3$.

Let $\psi\colon 2^\omega \to C(K)$ be continuous and take $g \in C(K)$ with $\psi^{-1}[B(h,\delta)+g] \neq \emptyset$. Then $\psi^{-1}[B(h,\delta)+g]$ contains a clopen $B \subseteq 2^\omega$.

Let $\sigma\colon 2^\omega \to B$ be a homeomorphism. Consider $\phi\colon 2^\omega \to \mathbb{R}$ given by $\phi(\alpha) = \psi(\sigma(\alpha))(0) - g(0)$. Note that ϕ is continuous and $x \in A \cap (x-\delta, x+\delta) \neq \emptyset$. Thus, there is $r \in \mathbb{R}$ with $\phi^{-1}[(A\cap(x-\delta,x+\delta))+r] \notin \mathcal{I}$.

We will show that $|r| \leq 2\delta + (x - h(0))$. Observe that:

$$\phi[2^\omega] = \{\psi(\alpha)(0) - g(0):\ \alpha \in B\} \subseteq (h(0)-\delta, h(0)+\delta)$$

(as $\psi[B] \subseteq B(h,\delta)+g$). Hence, for each $r' \in \mathbb{R}$ with $|r'| > 2\delta + (x-h(0))$ we have $\phi^{-1}[(A\cap(x-\delta,x+\delta))+r'] = \emptyset$ as $\phi[2^\omega]\cap((x-\delta,x+\delta)+r') = \emptyset$.

Let $g' \in C(K)$ be such that $g'(0) = g(0)+r$ and $\|g-g'\| \leq 2\delta + (x-h(0))$. Such g' exists by the above considerations. For each $\alpha \in B$ we have:

$$\|\psi(\alpha) - g' - h\| \leq \|\psi(\alpha) - g - h\| + \|g - g'\| < \delta + 2\delta + (x - h(0)) =$$

$$= (h(0)+\varepsilon - x) + (x - h(0)) = \varepsilon.$$

Finally, observe that $\psi^{-1}[Z_m + g'] \notin \mathcal{I}$ as

$$|\psi^{-1}[(B(h,\varepsilon)\cap A') + g']| \geq |\{\alpha \in B:\ \psi(\alpha)(0) - g'(0) \in A\cap(x-\delta,x+\delta)\}| =$$

$$= |\phi^{-1}[(A\cap(x-\delta,x+\delta))+r]|.$$

In the case of ℓ^∞ the proof is similar. □

3. Haar-1 sets

In this Section we study Haar-1 sets – we show that all countable subsets are Haar-1 and give an example of an uncountable Haar-1 subset of the real line.

Proposition 8. *Let X be an abelian Polish group. All countable subsets of X are Haar-1, i.e., $[X]^{\leq\omega} \subseteq \mathcal{H}1$.*

Proof. We will use [7, Theorem 1.1] stating that for an $\mathtt{F}_\sigma$ relation R on X with an R-independent set of size $\mathfrak{c}$ we can always find a Cantor R-independent set.

Let $A \subseteq X$ be countable. Define a relation $R \subseteq X \times X$ by:

$$(x, y) \in R \Leftrightarrow x - y \in A - A$$

for all $x, y \in X$. Note that it is reflexive and symmetric. Moreover, as $A - A$ is countable, it is easy to check that R is $\mathtt{F}_\sigma$.

We will show that there is an R-independent set (i.e., $(x, y) \notin R$ for each two distinct elements x and y of that set) of cardinality $\mathfrak{c}$. Indeed, by transfinite induction one can construct x_α, for $\alpha < \mathfrak{c}$, satisfying $x_\alpha \notin \bigcup_{\beta<\alpha}\{y \in X : (y, x_\beta) \in R\}$ (as $\{y \in X : (y, x_\beta) \in R\}$ is countable for each $\beta < \alpha$). Then $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ is the required set.

By [7, Theorem 1.1], there is a Cantor set $D \subseteq X$ with $(x, y) \notin R$ for all $x, y \in D$, $x \neq y$. We claim that D witnesses $A \in \mathcal{H}1$. Suppose to the contrary that $|(D - r) \cap A| > 1$ for some $r \in X$. Hence, there are $x, y \in D$, $x \neq y$, with $x - r, y - r \in A$. But then $(x, y) \in R$, since $(x - r) - (y - r) = x - y \in A - A$. A contradiction. □

Now we will give an example of a compact uncountable Haar-1 set. This will follow from the following more general result.

Proposition 9. *Let $A \subseteq \mathbb{R}$ be a closed set. If for each $\varepsilon > 0$ there is $r \in (0, \varepsilon)$ with $(A - r) \cap A = \emptyset$, then A is Haar-1.*

Proof. Construct a sequence $(d_n)_n$ of positive reals such that $\operatorname{dist}(A, A - d_n) > \sum_{m>n} d_m$, for each n. This can be done by applying our assumption. Indeed, start with any $d_0 \in (0, 1)$ with $(A - d_0) \cap A = \emptyset$. Once all d_m, for $m < n$, are already defined, pick

$$d_n \in \left(0, \min_{m<n}\left(\frac{\operatorname{dist}(A, A - d_m)}{2^{n-m}}\right)\right)$$

with $(A - d_n) \cap A = \emptyset$. It is easy to check that (d_n) is the desired sequence.

Define $D = \{\sum_{n\in\omega} x_n : x_n \in \{0, d_n\} \text{ for each } n\}$. Fix any $\sum_{n\in\omega} x_n, \sum_{n\in\omega} y_n \in D$, where $x_n, y_n \in \{0, d_n\}$, for each n. Let $k \in \omega$ be minimal with $x_k \neq y_k$.

Observe that:

$$\operatorname{dist}\left(\left(A - \sum_{n\leq k} x_n\right) \cap \left(A - \sum_{n\leq k} y_n\right)\right) = \operatorname{dist}\left((A - d_k) \cap A\right) > \left|\sum_{n>k} x_n - \sum_{n>k} y_n\right|.$$

Hence,

$$\left(A - \sum_{n\in\omega} x_n\right) \cap \left(A - \sum_{n\in\omega} y_n\right) = \emptyset$$

and D witnesses that A is Haar-1. □

Corollary 1. *Let $m \in \omega \setminus 4$. The set $A = \left\{\sum_{n\in\omega} \frac{i_n}{m^{n+1}} : \forall_{n\in\omega}\ i_n \in \{0, m-1\}\right\}$ is Haar-*1*.*

Proof. This follows from the previous Proposition, since $d_n = \frac{m-1}{2\cdot m^{n+1}}$ is a converging to 0 sequence with $(A - d_n) \cap A = \emptyset$ for all n. □

Remark 4. Actually, the above result can be transferred to the compact group $G = (\mathbb{Z}/m\mathbb{Z})^\omega$ (where $m \in \omega \setminus 4$) – it shows that the set $A' = \{(x_n)_n \in G : \forall_{n\in\omega}\ x_n \in \{0, m-1\}\}$ is Haar-1 in G.

4. Haar-2 sets

In this Section we distinguish $\overline{\mathcal{H}2}$ from $\mathcal{H}1$ and $\sigma\overline{\mathcal{H}1}$.

Notice that the next Theorem, besides providing a suitable example, explains why we had to assume $m > 3$ in Corollary 1.

Theorem 1. *The standard ternary Cantor set C is Haar-*2* but not Haar-*1*, i.e., $C \in \overline{\mathcal{H}2} \setminus \mathcal{H}1$.*

Proof. First we show that C is not Haar-1. Fix any Cantor set $D \subseteq \mathbb{R}$ and any $x, y \in D$ such that $d = x - y \in (0, 1)$. We will show that there is $c \in C \cap (C - d)$. It will follow that $c - y \in (C - x) \cap (C - y) \neq \emptyset$.

Let $(d_i)_i \subseteq 3^\omega$ be such that $d = \sum_{i\in\omega} \frac{d_i}{3^{i+1}}$ and put $T = \{j \in \omega : d_j = 1\}$. Define

$$c_i = \begin{cases} 0 & \text{if } |T \cap i| \text{ is even,} \\ 2 & \text{if } |T \cap i| \text{ is odd,} \end{cases}$$

$$c'_i = \begin{cases} d_i & \text{if } d_i \neq 1, \\ 0 & \text{if } d_i = 1 \text{ and } |T \cap i| \text{ is odd,} \\ 2 & \text{if } d_i = 1 \text{ and } |T \cap i| \text{ is even,} \end{cases}$$

and denote $c = \sum_{i\in\omega} \frac{c_i}{3^{i+1}}$ and $c' = \sum_{i\in\omega} \frac{c'_i}{3^{i+1}}$. Obviously, $c, c' \in C$.

We will show that $c = c' - d$. Let $(t_n)_n$ be an increasing enumeration of the set T. Firstly, observe that $c_i = 0$ and $c'_i - d_i = d_i - d_i = 0$ for all $i < t_0$. Thus, $\sum_{i<t_0} \frac{c_i}{3^{i+1}} = \sum_{i<t_0} \frac{c'_i}{3^{i+1}} - \sum_{i<t_0} \frac{d_i}{3^{i+1}}$. Fix now $n \in \omega$. We have:

$$\sum_{i\in[t_{2n}, t_{2n+2})} \frac{c_i}{3^{i+1}} = \frac{2}{3^{t_{2n}+2}} + \ldots + \frac{2}{3^{t_{2n+1}+1}} = \frac{1}{3^{t_{2n}+1}} - \frac{1}{3^{t_{2n+1}+1}} =$$

$$= \frac{2}{3^{t_{2n}+1}} - \frac{1}{3^{t_{2n}+1}} - \frac{1}{3^{t_{2n+1}+1}} = \sum_{i\in[t_{2n}, t_{2n+2})} \frac{c'_i}{3^{i+1}} - \sum_{i\in[t_{2n}, t_{2n+2})} \frac{d_i}{3^{i+1}}.$$

Thus, we see that $c = c' - d$.

Now we prove that C_1 is Haar-2. We start with some useful observations. Firstly, note that $C \cap (C - (\frac{1}{3^1} + \frac{1}{3^2})) \cap (C - \frac{2}{3^2}) = \emptyset$ (as $C \cap (C - (\frac{1}{3^1} + \frac{1}{3^2})) \subseteq [\frac{2}{9}, \frac{3}{9}]$ and $C \cap (C - \frac{2}{3^2}) \cap [\frac{2}{9}, \frac{3}{9}]$ is empty). Actually, we can generalize the above observation a little bit: we have $C \cap (C - ((\frac{1}{3^1} + \frac{1}{3^2}) + a')) \cap (C - (\frac{2}{3^2} + b')) = \emptyset$ for any $a', b' \in [0, \frac{1}{9})$ (as $\mathrm{dist}(C \cap (C - (\frac{1}{3^1} + \frac{1}{3^2})), (C - \frac{2}{3^2})) = \frac{1}{3^2}$).

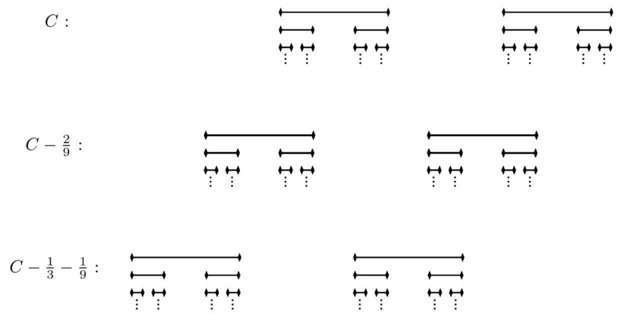

FIGURE 1

We can further generalize our observation: fix two sequences $\bar{a}=(a_i)_i, \bar{b}=(b_i)_i \in 3^k$ and denote $a=\sum_{i\in k}\frac{a_i}{3^{i+1}}$, $b=\sum_{i\in k}\frac{b_i}{3^{i+1}}$. Let α and β stand for the number of 1's in $\bar{a}$ and $\bar{b}$, respectively. Note first that $C\cap(C-a)$ is either finite (if α is odd) or equal to $C\cap I$ where I is some finite union of intervals of the form $[\frac{l}{3^k}, \frac{l+1}{3^k}]$, $l\in 3^k$ (if α is even). Consider first the case that both α and β are even. Then, similarly as in the beginning of this paragraph, for $x=\frac{1}{3^{k+1}}+\frac{1}{3^{k+2}}$ and $y=\frac{2}{3^{k+2}}$ we have $C\cap(C-(a+x+a'))\cap(C-(b+y+b'))=\emptyset$ for any $a',b'\in[0,\frac{1}{3^{k+2}})$.

Using the above observation we can see that in the general case (when α and β are arbitrary), by putting

$$\bar{x}=(x_i)_{i\in 11}=(1,1,1,1,1,0,1,1,1,1,1)$$

$$\bar{y}=(y_i)_{i\in 11}=(0,2,0,0,2,1,0,2,0,0,2)$$

and denoting $x=\sum_{i\in 11}\frac{x_i}{3^{k+1+i}}$ and $y=\sum_{i\in 11}\frac{y_i}{3^{k+1+i}}$, we have $C\cap(C-(a+x+a'))\cap(C-(b+y+b'))=\emptyset$ for any $a',b'\in[0,\frac{1}{3^{k+11}})$. Indeed, first two elements of $\bar{x}$ and $\bar{y}$ guarantee us that this intersection is empty when both α and β are even. If α is odd and β is even, then third elements of $\bar{x}$ and $\bar{y}$ lead us to the case that both $\alpha+|\{i\leq 3:\ x_i=1\}|$ (i.e., the number of 1's in $\bar{a}^\frown\bar{x}\restriction 3$) and $\beta+|\{i\leq 3:\ y_i=1\}|$ (i.e., the number of 1's in $\bar{b}^\frown\bar{y}\restriction 3$) are even and using fourth and fifth elements of $\bar{x}$ and $\bar{y}$ we see that the considered intersection is empty. The two remaining cases are similar.

We are ready to construct a Cantor set D witnessing that C is Haar-2. Define $k_0=0$, $k_1=1$ and $k_n=k_{n-1}+1+11\binom{2^n}{3}$ for $n>1$ (where $\binom{2^n}{3}$ denotes the number of 3-combinations of the set 2^n). We inductively pick finite sequences $d^s=(d^s_i)_i\in 3^{k_n-k_{n-1}}$ for every $s\in 2^n$ and $n\in\omega$. At the end D will consist of all points of the form $\sum_{n\in\omega}\bar{d}_{x|n}$ for some $x\in 2^\omega$ where $\bar{d}_\emptyset=0$ and $\bar{d}_s=\sum_{i=1}^{k_n-k_{n-1}}\frac{d^s_{i-1}}{3^{k_{n-1}+i}}$ for $s\in 2^n$. We start with $d^\emptyset=\emptyset$, $d^{(0)}=(0)$ and $d^{(1)}=(1)$. In the nth step define d^s for each $s\in 2^n$ in such a way that:

- d^s is a concatenation of (s_{n-1}) (the last element of the sequence s) and $\binom{2^n}{3}$ sequences each of which is equal to $\bar{x}$, $\bar{y}$ or $\bar{0}_{11}$, i.e., the sequence consisting of eleven 0's;

- for each $s, s', s'' \in 2^n$ with $s <_{lex} s' <_{lex} s''$, where $<_{lex}$ denotes the lexicographic order on $2^{<\omega}$ (note that there are exactly $\binom{2^n}{3}$ ways to choose s, s', s'') we can find $F = \{1 + 11p, \ldots, 1 + 11p + 10\}$ for some $p < \binom{2^n}{3}$ such that $(d_i^s)_{i\in F} = \bar{0}_{11}$, $(d_i^{s'})_{i\in F} = \bar{x}$ and $(d_i^{s''})_{i\in F} = \bar{y}$.

By the construction, given any $d, d', d'' \in D$ with $d < d' < d''$, we have

$$(C - d) \cap (C - d') \cap (C - d'') = C \cap (C - (d' - d)) \cap (C - (d'' - d')) = \emptyset.$$

It follows that $C \in \mathcal{H}2$. This finishes the proof. □

Remark 5. Theorem 1 can be transferred to the compact group $G = (\mathbb{Z}/3\mathbb{Z})^\omega$ – it shows that the set $A = \{(x_n)_n \in G : \forall_{n\in\omega}\ x_n \neq 1\}$ is Haar-2 but nor Haar-1 in G.

Corollary 2. *There is a compact Haar-2 set which is not a countable union of compact Haar-1 sets. Hence,* $\overline{\mathcal{H}2} \setminus \sigma\overline{\mathcal{H}1} \neq \emptyset$.

Proof. Consider the standard ternary Cantor set C. We already know that it is Haar-2. Observe that whenever $C \cap I \neq \emptyset$ for an open interval $I \subseteq \mathbb{R}$, we have $C \cap I \notin \mathcal{H}1$ (this can be done using the same methods as in the previous proof, since $C \cap I$ contains an isomorphic copy of C). It follows from Remark 3 that $C \notin \sigma\overline{\mathcal{H}1}$. □

5. Haar-n sets

Now we move to Haar-n sets for arbitrary $n \in \omega$. In this Section we give examples showing that the sequence $\mathcal{H}1 \subseteq \mathcal{H}2 \subseteq \ldots$ does not stabilize.

For any $m \in \omega \setminus \{0\}$ we denote:

$$C_m = \left\{ \sum_{k\in\omega} \frac{i_k}{(2m+1)^{k+1}} : \ i_k \in \{0, \ldots, 2m\} \setminus \{m\} \right\}.$$

Note that C_1 is the standard ternary Cantor set.

Lemma 1. *For every $m \in \omega \setminus \{0\}$ the set C_m is not Haar-$(m-1)$.*

Proof. Fix any Cantor set $D \subseteq \mathbb{R}$ and pick $x_0 < \ldots < x_{m-1} \in D$ such that $x_{m-1} - x_0 \in (0, \frac{1}{2m+1})$. Denote $d_i = x_i - x_0$ for $i = 1, \ldots, m-1$. We will show that there is $c \in C_m \cap \bigcap_{i=1}^{m-1}(C_m - d_i)$. It will follow that $c - x_0 \in \bigcap_{i=1}^{m-1}(C_m - x_i)$.

Let $(d_{i,j})_j \subseteq (2m+1)^\omega$ be such that $d_i = \sum_{j\in\omega} \frac{d_{i,j}}{(2m+1)^{j+1}}$, for each $i = 1, \ldots, m-1$ (note that $d_{i,0} = 0$ for all i by $x_0 < \ldots < x_{m-1}$ and $x_{m-1} - x_0 \in (0, \frac{1}{2m+1})$). We will inductively construct $(c_i)_i \in (2m+1)^\omega$. We start with $c_0 = 0$. Suppose now that c_i for all $i < n$ are already defined. Denote:

$$Z_n = \left\{ \sum_{k\in\omega} \frac{i_k}{(2m+1)^{k+1}} : \ \forall_k\ i_k \in 2m+1 \ \wedge\ \exists_k^\infty\ i_k \neq 2m \ \wedge\ i_n = m \right\}.$$

Pick such $c_n \in (2m+1) \setminus \{m+1\}$ that neither $\sum_{j=0}^n \frac{c_j}{(2m+1)^{j+1}}$ nor $\sum_{j=0}^n \frac{c_j}{(2m+1)^{j+1}} + \frac{1}{(2m+1)^{n+1}}$ belongs to the set

$$\bigcup_{i=1}^{m-1} \left(Z_n - \sum_{j=0}^n \frac{d_{i,j}}{(2m+1)^{j+1}} \right)$$

This is possible by the Pigeonhole Principle, since $|(2m+1)\setminus\{m+1\}|=2m$ and each set of the form $Z_n-\sum_{j=0}^{n}\frac{d_{i,j}}{(2m+1)^{j+1}}$ excludes at most two possible values of c_n.

Once the induction is completed, observe that $c=\sum_{j\in\omega}\frac{c_j}{(2m+1)^{j+1}}$ is an element of C_m. We will show that it is also an element of C_m-d_i for each $i=1,\ldots,m-1$. This will end the proof.

Fix $i\in\{1,\ldots,m-1\}$ and suppose to the contrary that $c\notin C_m-d_i$. Then $c\in\bigcup_{n\geq 1}(Z_n-d_i)$, since $c+d_i\in[0,1]$ (as $c_0=0$ and $d_i<\frac{1}{2m+1}$) and $[0,1]\setminus C_m=\bigcup_{n\geq 1}\mathrm{int}Z_n$. Thus, there is n_0 such that $c\in Z_{n_0}-d_i$. Observe that:

$$c\in\left[\sum_{j=0}^{n_0}\frac{c_j}{(2m+1)^{j+1}},\sum_{j=0}^{n_0}\frac{c_j}{(2m+1)^{j+1}}+\frac{1}{(2m+1)^{n_0+1}}\right]$$

and denote the above interval by I. By to choice of c_{n_0}, we know that endpoints of I do not belong to the set $Y=Z_{n_0}-\sum_{j=0}^{n_0}\frac{d_{i,j}}{(2m+1)^{j+1}}$, which is a union of intervals of the form $\left[\frac{p}{(2m+1)^{n_0+1}},\frac{p+1}{(2m+1)^{n_0+1}}\right)$, for some integer p. Thus, I and Y are disjoint. Moreover, the distance between $\{\max I\}$ and Y is at least $\frac{1}{(2m+1)^{n_0+1}}$. Now it suffices to observe that $\sum_{j>n_0}\frac{d_{i,j}}{(2m+1)^{j+1}}\in\left[0,\frac{1}{(2m+1)^{n_0+1}}\right)$. Hence, I is disjoint with $Z_{n_0}-d_i$, a contradiction. $\square$

Lemma 2. *For every $m\in\omega\setminus\{0\}$ the set C_m is Haar-$(2m+1)$.*

Proof. This proof is similar to the proof of Theorem 1.

At first fix any $k\in\omega$ and define

$$x_j=\frac{j}{(2m+1)^{k+1}}+\frac{2m-j}{(2m+1)^{k+2}},$$

for $j=0,\ldots,2m-1$,

$$x_{2m}=\frac{2m}{(2m+1)^{k+1}}+\frac{0}{(2m+1)^{k+2}}+\frac{1}{(2m+1)^{k+3}}$$

and $x_{2m+1}=0$. Denote

$$Z_k=\left\{\sum_{j\in\omega}\frac{i_j}{(2m+1)^{j+1}}:\ \forall_j\ i_j\in 2m+1\ \wedge\ i_k\neq m\right\}$$

and observe that $\bigcap_{j\in 2m+2}Z_k-x_j=\emptyset$ (since each gap in the set Z_k is of the form $\left(\frac{p}{(2m+1)^{k+1}},\frac{p+1}{(2m+1)^{k+1}}\right)$, for some positive integer p, and we simply cover the whole interval $\left[\frac{p+1}{(2m+1)^{k+1}}-\frac{1}{(2m+1)^{k}},\frac{p+1}{(2m+1)^{k+1}}\right)$ by translations of that gap). Thus, $\bigcap_{j\in 2m+2}C_m-x_j$ is empty as well, since $C_m\subseteq Z_k$.

We can further generalize this observation: for any $(a_{0,j})_j,\ldots,(a_{2m+1,j})_j\in(2m+1)^k$, if $a_i=\sum_{j\in k}\frac{a_{i,j}}{(2m+1)^{j+1}}$ for each $i=0,\ldots,2m+1$, then we have $\bigcap_{j\in 2m+2}C_m-(a_j+x_j)=\emptyset$ (since $\bigcap_{j\in 2m+2}C_m-a_j\subseteq\bigcap_{j\in 2m+2}Z_k-a_j$ and the latter intersection is equal to Z_k-a_0 intersected with some interval). Moreover, $\bigcap_{j\in 2m+2}C_m-(a_j+x_j+a'_j)=\emptyset$ for any $a'_0,\ldots,a'_{2m+1}\in[0,\frac{1}{(2m+1)^{k+3}})$ (since for each $i\in 2m+2$ the distance between $C_m-(a_i+x_i)$ and $\bigcap_{j\neq i}C_m-(a_j+x_j)$ is at least $\frac{1}{(2m+1)^{k+3}}$).

Now, using the above, we construct a Cantor set D witnessing that C_m is Haar-$(2m+1)$. Define $k_0 = 0$ and $k_n = k_{n-1} + 1 + 3\binom{2^n}{2m+2}$, for $n > 0$ (where $\binom{2^n}{2m+2}$ denotes the number of $(2m+2)$-combinations of the set 2^n; we put $\binom{2^n}{2m+2} = 0$ in the case $2^n < 2m+2$). Pick inductively finite sequences $\bar{d}^s = (d_i^s)_i \in (2m+1)^{k_n - k_{n-1}}$, for every $s \in 2^n$ and $n \in \omega$. At the end D will consist of all points of the form $\sum_{n\in\omega} d_{x|n}$ for some $x \in 2^\omega$, where $d_s = \sum_{i=1}^{k_n-k_{n-1}} \frac{d_{i-1}^s}{(2m+1)^{k_{n-1}+i}}$, for $s \in 2^n$. Start with $\bar{d}^\emptyset = \emptyset$. In the nth step define $\bar{d}^s$, for each $s = (s_i) \in 2^n$, in such a way that:

- $\bar{d}^s$ is a concatenation of (s_{n-1}) (the last element of the sequence s) and $\binom{2^n}{2m+2}$ sequences each of which is either $(0,0,0)$, $(2m,0,1)$ or $(j, 2m-j, 0)$, for some $j \in 2m$;
- for every $s_0, \ldots, s_{2m+1} \in 2^n$ (note that there are exactly $\binom{2^n}{2m+2}$ ways to choose $s_0, \ldots, s_{2m+1}$) we can find $F = \{1+3p, 2+3p, 3+3p\}$, for some $p < \binom{2^n}{2m+2}$, such that:

$$\{(\bar{d}_i^{s_j})_{i\in F}:\ j \in 2m+2\} = \{(0,0,0),(2m,0,1)\} \cup \{(j,2m-j,0):\ j \in 2m\}.$$

By the construction, given any $d_0, \ldots, d_{2m+1} \in D$, we have $\bigcap_{j=0}^{2m+1} C_m - d_j = \emptyset$. Hence, C_m is Haar-$(2m+1)$. $\square$

In the next Section we will need a slightly more general result:

Corollary 3. *For every $m \in \omega \setminus \{0\}$ the set $\bigcup_{k\in\mathbb{Z}} C_m + k$ is Haar-$(2m+1)$.*

Proof. This can be proved exactly the same way as Lemma 2. $\square$

As an immediate consequence of Lemmas 1 and 2 we get the following.

Theorem 2. *For every $m \in \omega \setminus \{0\}$ there is a compact Haar-$(2m+1)$ which is not Haar-$(m-1)$. Thus, $\overline{\mathcal{H}(2m+1)} \setminus \mathcal{H}(m-1) \neq \emptyset$ for each $m \in \omega \setminus \{0\}$.*

Remark 6. The above reasonings can be applied to the compact group $G = (\mathbb{Z}/(2m+1)\mathbb{Z})^\omega$ – it shows that the set $A = \{(x_n)_n \in G:\ \forall_{n\in\omega}\ x_n \neq m\}$ is Haar-$(2m+1)$ but nor Haar-$(m-1)$ in G.

Note that the above results are not sharp. Basing on experience gained in the previous Section, we suppose the following.

Conjecture 1. *For every $m \in \omega \setminus \{0\}$ the set C_m is Haar-$2m$ but not Haar-$(2m-1)$.*

Note that this is true for $m = 1$ by Theorem 1. We believe that the Conjecture can be proved using similar methods to the ones already used in this paper.

6. Haar-finite sets

Recall that each countable union of Haar-finite sets cannot be the whole space, as Haar-finite sets are Haar-null and Haar-meager. In this Section we show that a union of two Haar-finite sets does not have to be Haar-finite. This answers a question posed by Swaczyna during his talk on XLI Summer Symposium in Real Analysis (Wooster, 2017). Note also that, as a consequence of the next Theorem, $\mathcal{I}$ being an ideal does not guarantee that $\mathcal{HI}$ is an ideal.

Theorem 3. *The family of Haar-finite subsets of $\mathbb{R}$ is not an ideal.*

Proof. We need to construct $A, B \in \mathcal{H}\mathrm{Fin}$ and $X \notin \mathcal{H}\mathrm{Fin}$ with $A \cup B = X$. The proof consists of 5 steps.

Step 1: Construction of the set X.

Let $q_0 = 25$ and $q_n = q_{n-1} \cdot 25 \cdot 3^n$, for $n \geq 1$. Define a function $\phi\colon \omega^\omega \to \mathbb{R} \cup \{\infty\}$ by $\phi((x_i)_i) = \sum_{i\in\omega} \frac{x_i}{q_i}$. Put also $m_n = \frac{25\cdot 3^n - 1}{2}$, for all $n \in \omega$.

Define $L_0 = \{8, 10\}$ and let X_0 be the set of all points of the form $\phi((x_i)_i)$, where:

- $x_i \in 25 \cdot 3^i$, for all $i \in \omega$;
- $x_0 \notin L_0$;
- $x_i \notin [m_0 \cdot 3^i, (m_0 + 1) \cdot 3^i - 1] = \{m_0 \cdot 3^i + j :\ j = 0, \ldots, 3^i - 1\}$, for all $i \in \omega$.

Equivalently, $X_0 = C_{m_0} \setminus ((\frac{8}{25}, \frac{9}{25}) \cup (\frac{10}{25}, \frac{11}{25}))$. Observe that X_0 is Haar-$(2m_0 + 1)$ by Lemma 2.

For each $n \geq 1$ let $L_n = (3L_{n-1}) \cup (3L_{n-1} + 2)$. Notice that none L_n contains two consecutive integers. One can inductively show that $\max L_n < m_n - 1$, for all n. Indeed, $\max L_0 = 10 < 11 = m_0 - 1$ and we have:

$$\max L_{n+1} = 3 \cdot \max L_n + 2 < 3 \cdot \left(\frac{25 \cdot 3^n - 1}{2} - 1\right) + 2 < \frac{25 \cdot 3^{n+1} - 1}{2} - 1 = m_{n+1} - 1.$$

Moreover, $\min L_n > \frac{m_n + 1}{2}$, for all n, as $\min L_0 = 8 > \frac{13}{2} = \frac{m_0 + 1}{2}$ and:

$$\min L_{n+1} = 3 \cdot \min L_n > 3 \cdot \frac{\frac{25\cdot 3^n - 1}{2} + 1}{2} > \frac{\frac{25\cdot 3^{n+1} - 1}{2} + 1}{2} > \frac{m_{n+1} + 1}{2}.$$

In particular, $\frac{m_n}{2} < \min L_n < \max L_n < m_n$ for all n.

For each n let X_n consist of all points of the form $\phi((x_i)_i)$, where:

- $x_i \in 25 \cdot 3^i$, for all i;
- $x_m \in L_m$, for all $m < n$;
- $x_n \notin L_n$;
- $x_i \notin [m_n \cdot 3^{i-n}, (m_n + 1) \cdot 3^{i-n} - 1]$, for all $i \geq n$.

Equivalently, $X_n \subseteq \bigcup_{l \in L_{n-1}} \left[\frac{l}{q_{n-1}}, \frac{l+1}{q_{n-1}}\right]$ and for each $l \in L_{n-1}$ the intersection $X_n \cap \left[\frac{l}{q_{n-1}}, \frac{l+1}{q_{n-1}}\right]$ is equal to the linear copy of C_{m_n} (via the linear bijection between $[0,1]$ and $\left[\frac{l}{q_{n-1}}, \frac{l+1}{q_{n-1}}\right]$) with the elements of $\bigcup_{l' \in L_n} \left(\frac{l'}{q_n}, \frac{l'+1}{q_n}\right)$ removed.

Observe that, by Corollary 3, X_n is Haar-$(2m_n + 1)$ as a subset of a linear copy of $\bigcup_{i<q_{n-1}} (C_{m_n} + i)$ via the linear bijection between $[0, q_{n-1}]$ and $[0,1]$. Actually, this reasoning even shows that $\bigcup_{i\leq n} X_i$ is Haar-$(2m_n + 1)$ (as each X_i, with $i \leq n$, is a subset of the mentioned linear copy).

The set X is given by:

$$X = \left(\bigcup_{n\in\omega} X_n\right) \cup \{\phi((x_i)_i) :\ \forall_{i\in\omega}\ x_i \in L_i\}.$$

Observe that X is compact.

Step 2: X is not Haar-finite.

Before proving that X is not Haar-finite, let us construct some auxiliary sets T_n^k, for $k, n \in \omega$.

For each $k \in \omega$ define T_0^k as the set of all points of the form $\phi((x_i)_i)$, where:

- $x_i \in 25 \cdot 3^i$ for all $i \in \omega$;
- $x_i \neq 25 \cdot 3^i - 1$ for infinitely many $i \in \omega$;
- $x_k \in \{m_0 \cdot 3^k + j : \ j = 0, \ldots, 3^k - 1\}$.

For each $k \in \omega$ and $n \in \omega \setminus \{0\}$ let T_n^k consist of all points of the form $\phi((x_i)_i)$, where:

- $x_i \in 25 \cdot 3^i$ for all $i \in \omega$;
- $x_i \neq 25 \cdot 3^i - 1$ for infinitely many $i \in \omega$;
- $x_i \in L_i$ for all $i < n$;
- $x_k \in [m_n \cdot 3^{k-n}, (m_n + 1) \cdot 3^{k-n} - 1]$.

Define $T_n = \bigcup_{k \in \omega} T_n^k$ for all n. Notice that $[0,1] \setminus T_n \subseteq X$ for each n.

Now we proceed to proving that X is not Haar-finite. Fix any Cantor set $C \subseteq \mathbb{R}$ and choose a decreasing sequence $(c_i')_i \subseteq C$ such that $c_0 = c_0' - \inf C < \frac{12}{25} = \frac{m_0}{q_0}$ and $c_i = c_i' - \inf C < \frac{1}{q_{i-1}}$ for all $i \in \omega$, $i \geq 1$.

Let $(c_{i,j})_j$ be such that $c_i = \phi((c_{i,j})_j)$, for each $i \in \omega$. We will inductively construct a sequence $(r_i)_i$ with $r_i \in L_i$, for each $i \in \omega$. We start by picking $r_0 \in L_0$ such that neither $\frac{r_0}{q_0}$ nor $\frac{r_0+1}{q_0}$ belongs to the set $T_0^0 - \frac{c_{0,0}}{q_0}$. This is possible by the Pigeonhole Principle as $T_0^0 - \frac{c_{0,0}}{q_0}$ (which is an interval of the form $\left[\frac{p}{q_0}, \frac{p+1}{q_0}\right)$ for some integer p) can exclude at most two consecutive values of $r_0 \in q_0$, i.e., at most one element of $L_0 = \{8, 10\}$. Next, pick $r_1 \in L_1$ such that neither $\sum_{j=0}^{1} \frac{r_j}{q_j}$ nor $\sum_{j=0}^{1} \frac{r_j}{q_j} + \frac{1}{q_1}$ belongs to the set

$$\left(T_0^1 - \sum_{j=0}^{1} \frac{c_{0,j}}{q_j}\right) \cup \left(T_1^1 - \sum_{j=0}^{1} \frac{c_{1,j}}{q_j}\right).$$

Again, this is possible by the Pigeonhole Principle as $L_1 = \{24, 26, 30, 32\}$ and $T_0^1 - \sum_{j=0}^{1} \frac{c_{0,j}}{q_j}$ can exclude at most four consecutive values of $r_1 \in 25 \cdot 3$ (i.e., it cannot exclude simultaneously something from $\{24, 26\}$ and something from $\{30, 32\}$) while $T_1^1 - \sum_{j=0}^{1} \frac{c_{1,j}}{q_j}$ can exclude at most two consecutive values of $r_1 \in 25 \cdot 3$. Suppose now that r_i, for all $i < k$, are already defined. There is some $r_k \in L_k$ such that neither $\sum_{j=0}^{k} \frac{r_j}{q_j}$ nor $\sum_{j=0}^{k} \frac{r_j}{q_j} + \frac{1}{q_k}$ belongs to the set

$$\bigcup_{i=0}^{k} \left(T_i^k - \sum_{j=0}^{k} \frac{c_{i,j}}{q_j}\right).$$

Similarly as above, one can show that $T_0^k - \sum_{j=0}^{k} \frac{c_{0,j}}{q_j}$ excludes at most half of the set L_k, $T_1^k - \sum_{j=0}^{k} \frac{c_{1,j}}{q_j}$ excludes at most one-fourth of the set L_k etc.

Once the induction is completed, observe that $r = \phi((r_i)_i)$ is an element of X. What is more, we will show that it belongs to $\bigcap_{i \in \omega}(X - c_i)$.

Fix $i \in \omega$. We will show that $r \notin T_i - c_i$. As $r + c_i \leq r + c_0 < \frac{11}{25} + \frac{12}{25} < 1$ (recall that $r_0 \in L_0 = \{8, 10\}$) and $[0,1] \setminus T_i \subseteq X$, this will finish this step.

Suppose to the contrary that $r \in T_i - c_i$. Thus, there is k_0 such that $r \in T_i^{k_0} - c_i$. Observe that:

$$r \in \left[\sum_{j=0}^{k_0} \frac{r_j}{q_j}, \sum_{j=0}^{k_0} \frac{r_j}{q_j} + \frac{1}{q_{k_0}}\right].$$

Denote the above interval by I. By to choice of r_{k_0}, we know that endpoints of I do not belong to the set $T_i^{k_0} - \sum_{j=0}^{k_0} \frac{c_{i,j}}{q_j}$, which is a union of intervals of the form $\left[\frac{p}{q_{k_0}}, \frac{p+1}{q_{k_0}}\right)$, for some integer p. Thus, the distance between $\{\max I\}$ and $T_i^{k_0} - \sum_{j=0}^{k_0} \frac{c_{i,j}}{q_j}$ is at least $\frac{1}{q_{k_0}}$. Now it suffices to observe that $\sum_{j>k_0} \frac{c_{i,j}}{q_j} < \frac{1}{q_{k_0}}$. Hence, I is disjoint with $T_i^{k_0} - \sum_{j=0}^{k_0} \frac{c_{i,j}}{q_j}$, a contradiction.

Step 3: Partition of X into two sets A and B.

Let B_0 consist of those $\phi((x_i)_i)$ belonging to X_0 such that

$$\mathrm{dist}\left(\{x_0 - \lfloor\frac{m_0}{2}\rfloor\}, L_0\right) \leq 1.$$

Equivalently, $B_0 = X_0 \cap \left[\frac{7}{13}, \frac{11}{13}\right]$.

For each $n \in \omega \setminus \{0\}$ let B_n consist of those $\phi((x_i)_i)$ belonging to X_n such that:

$$\mathrm{dist}\left(\{x_n - \lfloor\frac{m_n}{2}\rfloor\}, L_n\right) \leq 1$$

as well as those $\phi((x_i)_i)$ belonging to X_n such that:

$$\mathrm{dist}\left(\{x_n + \lfloor\frac{m_n}{2}\rfloor\}, L_n\right) \leq 1.$$

Define $B = \bigcup_{n\in\omega} B_n$ and $A = \overline{X \setminus B}$. Observe that B is $\mathtt{F}_\sigma$ while A is compact. Moreover, $A \setminus (X \setminus B)$ consists of countably many points – endpoints of intervals used in the definitions of sets B_n.

Step 4: A is Haar-finite.

We will construct a Cantor set D witnessing that A is Haar-finite.

Define $k_0 = 0$ and $k_{n+1} = k_n + 1 + 3\binom{2^n}{2f_n+2}$, for $n > 1$, where $(f_n)_n$ is any non-decreasing sequence with $\lim_{n\to\infty} f_n = \infty$ and $2f_n + 2 < 2^n$. Pick inductively finite sequences $\bar{d}^s = (d_i^s)_i \in \omega^{k_n - k_{n-1}}$, for every $s \in 2^n$ and $n \in \omega$. At the end D will consist of all points of the form $\sum_{n\in\omega} d_{x|n}$ for some $x \in 2^\omega$, where

$$d_s = \sum_{i=1}^{k_n - k_{n-1}} \frac{d_{i-1}^s}{q_{k_{n-1}+i-1}}$$

for $s \in 2^n$. Start with $\bar{d}^\emptyset = \emptyset$. In the nth step define $\bar{d}^s$, for each $s = (s_i) \in 2^n$, in such a way that:

- $\bar{d}^s$ is a concatenation of $(s_{n-1} \cdot \lfloor\frac{m_{k_{n-1}}}{2}\rfloor)$ and $\binom{2^n}{2f(n)+2}$ sequences each of which is either $(0,0,0)$, $(2f_n, 0, 1)$ or $(j, 2f_n - j, 0)$, for some $j \in 2f(n)$;
- for every $s_0, \ldots, s_{2f(n)+1} \in 2^n$ we can find $F = \{1+2p, 2+2p, 3+2p\}$, for some $p < \binom{2^n}{2f(n)+2}$, such that:

$$\left\{(\bar{d}_i^{s_j})_{i\in F} : \ j \in 2f(n)+2\right\} = \{(0,0,0), (2f_n, 0, 1)\} \cup \{(j, 2f_n - j, 0) : \ j \in 2f_n\}.$$

Now we will show that D is as needed. Given any infinite $\{x_j : \ j \in \omega\} \subseteq 2^\omega$, denote $d_j = \sum_{n\in\omega} d_{x_j|n} \in D$, for each $j \in \omega$. Our goal is to prove that $\bigcap_{j\in\omega}(A - d_j) = \emptyset$.

Let $\tilde{n}$ be minimal such that $x_{j_0}|\tilde{n} \neq x_{j_1}|\tilde{n}$ for some $j_0, j_1 \in \omega$. Without loss of generality we can assume that $x_j|\tilde{n} = x_{j_1}|\tilde{n}$ for infinitely many $j \in \omega$. The crucial observation is the following:

$$(A-d_{j_0})\cap(A-d_j)=\left(\bigcup_{l\le k_{\tilde n-1}+1}X_l-d_{j_0}\right)\cap\left(\bigcup_{l\le k_{\tilde n-1}+1}X_l-d_j\right),$$

whenever $x_j|\tilde n = x_{j_1}|\tilde n$ (by the definition of A and choice of first coordinates of $\bar d^{x_{j_0}|\tilde n}$ and $\bar d^{x_{j_1}|\tilde n}$). Denote $\tilde k = k_{\tilde n-1}+1$ and recall that $\bigcup_{i\le\tilde k}X_i$ is Haar-$(2m_{\tilde k}+1)$. Let $j_2,\ldots,j_{2m_{\tilde k}+2}$ be such that $x_{j_i}|\tilde n = x_{j_1}|\tilde n$ for $i=1,\ldots,2m_{\tilde k}+2$. Then, by the construction of all $\bar d^s$, we get that $\bigcap_{i=1}^{2m_{\tilde k}+2}\left(\bigcup_{l\le\tilde k}X_l-d_{j_i}\right)=\emptyset$ (see the proof of Lemma 1 for details). Thus

$$\bigcap_{i\le 2m_{\tilde k}+2}(A-d_{j_i})=\bigcap_{i=1}^{2m_{\tilde k}+2}\left((A-d_{j_i})\cap(A-d_{j_0})\right)=$$

$$=\bigcap_{i=1}^{2m_{\tilde k}+2}\left(\left(\bigcup_{l\le\tilde k}X_l-d_{j_i}\right)\cap\left(\bigcup_{l\le\tilde k}X_l-d_{j_0}\right)\right)\subseteq\bigcap_{i=1}^{2m_{\tilde k}+2}\left(\bigcup_{l\le\tilde k}X_l-d_{j_i}\right)=\emptyset.$$

Step 5: B is Haar-finite.

We will construct a Cantor set E witnessing that B is Haar-finite.

Define $m_0=0$ and $m_{n+1}=m_n+2+3\binom{2^n}{2f_n+2}$, for $n>1$, where $(f_n)_n$ is as in the previous step. Inductively pick finite sequences $\bar e^s=(e^s_i)\in\omega^{m_n-m_{n-1}}$, for every $s\in 2^n$ and $n\in\omega$: $\bar e^\emptyset=\emptyset$ and $\bar e^s$, for each $s=(s_i)\in 2^n$, is such that:

- $\bar e^s$ is a concatenation of $(s_{n-1},0)$ and $\binom{2^n}{2f(n)+2}$ sequences each of which is either $(0,0,0)$, $(2f_n,0,1)$ or $(j,2f_n-j,0)$, for some $j\in 2f(n)$;
- for every $s_0,\ldots,s_{2f(n)+1}\in 2^n$ we can find $F=\{1+2p,2+2p,3+2p\}$, for some $p<\binom{2^n}{2f(n)+2}$, such that:

$$\left\{(\bar e^{s_j}_i)_{i\in F}:\ j\in 2f(n)+2\right\}=\{(0,0,0),(2f_n,0,1)\}\cup\{(j,2f_n-j,0):\ j\in 2f_n\}.$$

Let E consist of all points of the form $\sum_{n\in\omega}e_{x|n}$, for some $x\in 2^\omega$, where $e_s=\sum_{i=1}^{m_n-m_{n-1}}\frac{e^s_{i-1}}{q_{k_{n-1}+i-1}}$, for $s\in 2^n$.

Now, given any infinite $\{(x_j):\ j\in\omega\}\subseteq 2^\omega$, denote $e_j=\sum_{n\in\omega}e_{x_j|n}\in E$, for each $j\in\omega$. Define $\tilde n$, j_0 and j_1 similarly as in the previous step. Again, we have:

$$(B-e_{j_0})\cap(B-e_j)=\left(\bigcup_{l\le m_{\tilde n-1}+1}X_l-e_{j_0}\right)\cap\left(\bigcup_{l\le m_{\tilde n-1}+1}X_l-e_j\right),$$

whenever $x_j|\tilde n = x_{j_1}|\tilde n$ (here we use the fact that none L_n contains two consecutive integers). The rest of the proof is exactly the same as in the previous step.

This finishes the entire proof. □

Remark 7. Note that the above construction can be also conducted in some countable product of finite groups – namely, in the compact group $\prod_{n\in\omega}G_n$, where $(G_n)_n$ is a sequence of finite groups with $|G_0|=q_0$ and $|G_n|=q_n/q_{n-1}$ ($(q_n)_n$ is as in the previous proof).

The above result can be transferred to the case of null-finite sets. Recall that a subset A of an abelian Polish group X is null-finite if there is a convergent sequence $(x_n) \subseteq X$ such that $\{n \in \omega : \ x_n - x \in A\}$ is finite for all $x \in X$. Clearly, each Haar-finite set is null-finite (a witness for a null-finite set is compact, but is does not have to be uncountable as in the case of Haar-finite sets). Moreover, it is easy to observe that for $A \neq \emptyset$ the witnessing sequence $(x_n) \subseteq X$ has to have infinitely many values.

The question whether Borel null-finite sets form an ideal was posed in the first version of [2] and asked by Banakh during his talk at the conference Frontiers of Selection Principles (Warsaw, 2017). It should be mentioned that each non-discrete metric abelian group X is a union of two null-finite sets. Thus, the restriction to Borel sets in the above question is crucial.

Corollary 4. *The family of null-finite subsets of* $\mathbb{R}$ *is not an ideal.*

Proof. It suffices to consider the set $Y = X \cup -X$, where X is as in the previous proof. Then Y is a union of four Haar-finite (so also null-finite) sets: A, B, $-A$ and $-B$. However, one can show that Y is not null-finite by setting any injective convergent sequence and considering two cases: either it has a decreasing subsequence (in this case the proof is exactly the same as above) or an increasing one (in this case $-X$ is used). □

It is easy to construct a Haar-finite subset of $\mathbb{R}$ which is not Haar-n for any $n \in \omega$ – it suffices to consider $\bigcup_{m\in\omega} C_m + 2m$. Now we establish a compact example.

Corollary 5. *There is a compact Haar-finite subset of* $\mathbb{R}$ *which is not Haar-n for any* $n \in \omega$. *Therefore,* $\overline{\mathcal{H}Fin} \setminus \bigcup_{n\in\omega} \mathcal{H}n \neq \emptyset$.

Proof. It suffices to consider the set A from the proof of Theorem 3. We already know that it is compact and Haar-finite. To show that it is not Haar-n, for any $n \in \omega$, observe that A consists an isometric copy of C_m for arbitrarily big m. Indeed, $\overline{X_n \setminus B} \subseteq \overline{X \setminus B} = A$, for each n, and if $I \cap (\overline{X_n \setminus B}) \neq \emptyset$ for some open interval I, then $I \cap (\overline{X_n \setminus B})$ contains an isometric copy of $C_{m_n} \notin \mathcal{H}(m_n - 1)$. Therefore, it suffices to prove the non-emptiness of $\overline{X_n \setminus B}$.

Notice that:

$$B \cap X_n = \Big\{\phi((x_i)_i) \in X_n : \ \mathrm{dist}\left(\{x_n - \lfloor \frac{m_n}{2} \rfloor\}, L_n\right) \leq 1 \ \vee \\ \vee \ \mathrm{dist}\left(\{x_n + \lfloor \frac{m_n}{2} \rfloor\}, L_n\right) \leq 1\Big\}.$$

Thus, if $\phi((x_i)_i) \in B \cap X_n$, then $x_n \leq \max L_n + \lfloor \frac{m_n}{2} \rfloor + 1 < \frac{3}{2} m_n$ (the last inequality is due to $\max L_n < m_n - 1$). Hence,

$$\{\phi((x_i)_i) \in X_n : \ x_n = 2m_n\} \subseteq \overline{X_n \setminus B} \subseteq A$$

and we are done. □

Observe that the above example is a countable union of Haar-n sets. A natural question is whether this is the case for all Haar-finite sets. Actually, we do not know any example denying that observation.

Question 1. Is it true that $\mathcal{H}\mathrm{Fin} \subseteq \sigma\left(\bigcup_{n\in\omega} \mathcal{H}n\right)$, i.e., is each Haar-finite set a countable union of Haar-n sets?

Although, we have a counterexample for the opposite inclusion.

Corollary 6. *A countable union of compact Haar-n subset of* $\mathbb{R}$ *does not have to be Haar-finite, i.e.,* $\sigma\left(\bigcup_{n\in\omega}\overline{\mathcal{H}n}\right)\setminus\mathcal{H}Fin\neq\emptyset$.

Proof. It suffices to consider the set X from the proof of Theorem 3. It is not Haar-finite. However, it is a countable union of sets $X_n\in\mathcal{H}(2m_n+1)$, for $n\in\omega$, and the set $\{\phi((x_i)_i):\ \forall_{i\in\omega}\ x_i\in L_i\}$, which is Haar-1 by Proposition 9, as witnessed by the sequence $(\frac{1}{q_n})_n$ (recall that none L_n contains two consecutive integers). □

Question 2. Is it true that a union of two Haar-n sets is Haar-finite? What about compact Haar-n sets?

7. Haar-countable sets

In this Section we construct a Haar-countable set which is not Haar-finite and a Haar-null and Haar-meager set which is not a countable union of Haar-countable sets. Also, we give a partial answer to a question posed in [2] concerning countable unions of null-finite sets.

Theorem 4. *There is a compact Haar-countable subset of* $\mathbb{R}$ *which is not Haar-finite. Thus,* $\overline{\mathcal{H}Ctbl}\setminus\mathcal{H}Fin\neq\emptyset$.

Proof. Consider the set X from the proof of Theorem 3. We have already shown that it is not Haar-finite. Now we will show that the Cantor set E constructed in the last step of the proof of Theorem 3 witnesses that X is Haar-countable.

Take any uncountable set $\{x^\alpha=(x_i^\alpha)_i:\ \alpha<\omega_1\}\subseteq 2^\omega$ and denote $e_\alpha=\sum_{n\in\omega}\bar{e}_{x^\alpha|n}\in E$, for each $\alpha<\omega_1$. Suppose to the contrary that $z\in\bigcap_{\alpha<\omega_1}(X-e_\alpha)$ and let $m\in\omega$ be minimal such that $x_m^\alpha=0$ for uncountably many $\alpha<\omega_1$ and $x_m^\alpha=1$ for uncountably many $\alpha<\omega_1$.

Observe first that:

$$(\{\phi(x_i):\ \forall_{i\in\omega}\ x_i\in L_i\}-e_{\alpha_0})\cap(\{\phi(x_i):\ \forall_{i\in\omega}\ x_i\in L_i\}-e_{\alpha_1})=0$$

provided that $x_m^{\alpha_0}=0$ and $x_m^{\alpha_1}=1$ (by the fact that none L_n contains two consecutive integers). Thus, $z\in\bigcup_{n\in\omega}X_n-e_\alpha$ either for all α with $x_m^\alpha=0$ or for all α with $x_m^\alpha=1$. Now, as there are uncountably many such α, we conclude that there is k with $z\in X_k-e_\alpha$ for uncountably many α. However, by the definition of E, this is impossible, since $\bigcap_{i\leq 2m_k+1}X_k-e_{\alpha_i}=\emptyset$ for any pairwise distinct $\alpha_0,\ldots,\alpha_{2m_k+1}<\omega_1$. This finishes the proof. □

Theorem 5. *There is a null and meager compact subset of* $\mathbb{R}$ *which is not Haar-countable. Thus,* $\overline{\mathcal{HN}}\cap\overline{\mathcal{HM}}\setminus\mathcal{H}Ctbl\neq\emptyset$.

Proof. Fix an increasing sequence $(k_n)_n$ of integers with $k_0=0$. Let $q_0=3$ and $q_i=(2n+3)q_{i-1}$ whenever $k_n\leq i<k_{n+1}$. Define $f\colon\omega^\omega\to\mathbb{R}\cup\{\infty\}$ by $f((x_i)_i)=\sum_{i\in\omega}\frac{x_i}{q_i}$. Denote by X the set consisting of all reals of the form $f((x_i)_i)$, where $x_i\in(2n+3)\setminus\{n+1\}$ whenever $k_n\leq i<k_{n+1}$. Obviously, X is nowhere dense. Moreover, if $(k_n)_n$increases sufficiently fast, it is also null.

In order to show that X is not Haar-countable, fix any Cantor set D. Without loss of generality we may assume that $\inf D=0$ and $\sup D\leq\frac{2}{3}$ (by translating D and intersecting it with the interval $[0,\frac{2}{3}]$). Let $\{d_s:\ s\in 2^{<\omega}\}\subseteq\mathbb{R}$ be such that

$$D=\left\{\sum_{n\in\omega}d_{\alpha|n}:\ \alpha\in 2^\omega\right\}$$

and $d_{s^\frown(1)} > \sum_{i\geq 2} d_{s^\frown\bar{1}_i}$, for each $s \in S$, where $\bar{1}_i = (1,\ldots,1) \in 2^i$. We can additionally assume that $d_s = 0$ for all $s \notin S = \{s^\frown(1) : s \in 2^{<\omega}\}$ (i.e., S is the set of all finite $0-1$ sequences with 1 at the end).

We inductively pick $t_s \in S$, for all $s \in S$. Let $t_{(1)}$ be of the form $(0,\ldots,0,1)$ and such that $d_{t_{(1)}} < \frac{1}{2q_{k_3}}$. Suppose now that t_s, for all $s \in S \cap 2^{<n}$, are already defined. Find minimal $m \in \omega$ with $2^{n-1} < \frac{m}{2}$. For each $s \in 2^{n-1}$ let $t_{s^\frown(1)}$ be such that:

(a) $d_{t_{s^\frown(1)}} < \frac{1}{2q_{k_{m+1}}}$;
(b) if $s = (0,\ldots,0)$, then $t_{s^\frown(1)}$ is of the form $(0,\ldots,0,1)$;
(c) if $s' \in S$ is the longest sequence such that $s' \subseteq s$, then $t_{s^\frown(1)}$ is a concatenation of $t_{s'}$ and some sequence of the form $(0,\ldots,0,1)$.

Let also $t_s = \emptyset$ for $s \in 2^\omega \setminus S$ (i.e., $d_{t_s} = 0$ for $s \in 2^\omega \setminus S$) and define:

$$E = \left\{ \sum_{n\in\omega} d_{t_{\alpha|n}} : \alpha \in 2^\omega \right\}.$$

Obviously, E is uncountable. By conditions (b) and (c) we also have $E \subseteq D$. Indeed, given any $\alpha \in 2^\omega$ define $L = \{n \in \omega : \alpha|n \in S\}$ and let $(l_i)_i$ be an increasing enumeration of L. Then for $\alpha' = \bigcup_{i\in\omega} t_{\alpha|l_i}$ (i.e., $\alpha'|\mathrm{lh}(t_{\alpha|l_i}) = t_{\alpha|l_i}$ for each i) we get $\sum_{n\in\omega} d_{t_{\alpha|n}} = \sum_{n\in\omega} d_{\alpha'|n} \in D$.

It remains to show that $\bigcap_{e\in E}(X-e) \neq \emptyset$. Let $e_\alpha = \sum_{n\in\omega} d_{t_{\alpha|n}}$ and $(e_i^\alpha)_i$ be such that $e_\alpha = f(e_i^\alpha)$, for $\alpha \in 2^\omega$.

We will inductively construct a sequence $(x_i)_i \in \omega^\omega$. We start with $x_0 = 0$. Suppose now that x_j, for all $j < i$, are already defined. Let m be such that $k_m \leq i < k_{m+1}$ and define Z_i as the set consisting of all reals of the form $f((y_j)_j)$, where $y_j \in 2n+3$ whenever $k_n \leq j < k_{n+1}$, $y_j \neq 2n+2$ for infinitely many $j \in \omega$, and $y_i = m+1$. Pick $x_i \in (2m+3) \setminus (m+2)$ in such a way that neither $\sum_{j\leq i} \frac{x_j}{q_j}$ nor $\sum_{j\leq i} \frac{x_j}{q_j} + \frac{1}{q_i}$ belongs to the set

$$\bigcup_{\alpha<2^\omega} \left(Z_i - \sum_{j\leq i} \frac{e_j^\alpha}{q_j} \right).$$

This is possible by the Pigeonhole Principle. Indeed, observe first that $|(2m+3)\setminus(m+2)| = m+1$ and each set of the form $Z_i - \sum_{j\leq i} \frac{e_j^\alpha}{q_j}$ excludes at most two values of x_i. We will show that there are at most $\frac{m}{2}$ pairwise distinct sets of the form $Z_i - \sum_{j\leq i} \frac{e_j^\alpha}{q_j}$. Let n be maximal such that $2^{n-1} < \frac{m}{2}$. By condition (a) we have $\sum_{j\leq i} \frac{e_j^\alpha}{q_j} = \sum_{j\leq i} \frac{e_j^\beta}{q_j}$ whenever $\alpha|n = \beta|n$. Thus,

$$\left\{ \sum_{j\leq i} \frac{e_j^\alpha}{q_j} : \alpha < 2^\omega \right\} = \left\{ \sum_{j\leq i} \frac{e_j^\alpha}{q_j} : \alpha = s^\frown(0,0,\ldots) \text{ for some } s \in 2^{n-1} \right\}.$$

Now it suffices to observe that the latter set has cardinality at most $2^{n-1} < \frac{m}{2}$.

To finish the proof, notice that $x = f((x_i)_i)$ belongs to $\bigcap_{\alpha<2^\omega}(X - e_\alpha)$. Indeed, fix any α and suppose to the contrary that $x \notin X - e_\alpha$. Then $x \in \bigcup_{i\in\omega} Z_i - e_\alpha$, since $[0,1]\setminus X \subseteq \bigcup_{i\in\omega} Z_i$ and $x + e_\alpha \leq x + \sup D \leq \frac{1}{3} + \frac{2}{3} = 1$ (recall that $x_0 = 0$).

Thus, there is i_0 such that $x \in Z_{i_0} - e_\alpha$. Observe that

$$x \in \left[\sum_{j\le i_0} \frac{x_j}{q_j}, \sum_{j\le i_0} \frac{x_j}{q_j} + \frac{1}{q_{i_0}}\right].$$

Denote the above interval by I. By to choice of x_{i_0}, we know that endpoints of I do not belong to the set $Z_{i_0} - \sum_{j\le i_0} \frac{e_j^\alpha}{q_j}$, which is a union of intervals of the form $[\frac{p}{q_{i_0}}, \frac{p+1}{q_{i_0}})$,for some integer p. Thus, the distance between $\max I$ and $Z_{i_0} - \sum_{j\le i_0} \frac{e_j^\alpha}{q_j}$ is at least $\frac{1}{q_{i_0}}$. Now it suffices to observe that $\sum_{j>i_0} \frac{e_j^\alpha}{q_j} < \frac{1}{q_{i_0}}$. Hence, I is disjoint with $Z_{i_0} - \sum_{j\le i_0} \frac{e_j^\alpha}{q_j}$, a contradiction. □

Remark 8. The above construction can also be conducted in the compact group $\prod_{n\in\omega} G_n$, where $(G_n)_n$ is a sequence of finite groups with $|G_0| = q_0$ and $|G_n| = q_n/q_{n-1}$ ($(q_n)_n$ is as in the previous proof).

Trivially, a countable union of Haar-countable sets is Haar-null and Haar-meager. Now we show that the opposite inclusion fails at least in the case of compact Haar-countable sets.

Corollary 7. *There is a compact null and meager subset of* $\mathbb{R}$ *which is not a countable union of compact Haar-countable sets. Thus,* $\overline{\mathcal{HN}} \cap \overline{\mathcal{HM}} \setminus \sigma\overline{\mathcal{H}Ctbl} \neq \emptyset$.

Proof. Consider the set X from the previous proof. We already know that it is null and meager. Moreover, $X \cap I \neq \emptyset$, for an open interval I, implies $X \cap I \notin \overline{\mathcal{H}\text{Ctbl}}$ (as $X \cap I$ contains an isometric copy of X). Hence, $X \notin \sigma\overline{\mathcal{H}\text{Ctbl}}$ (see Remark 3). □

Next Corollary gives a partial answer to the following problem posed in [2]: is $\mathcal{HN} \cap \mathcal{HM}$ equal to the σ-ideal generated by null-finite sets?

Corollary 8. *There is a compact null and meager subset of* $\mathbb{R}$ *which is not a countable union of compact null-finite sets.*

Proof. Let X be as in the proof of Theorem 5. We will show that X is not null-finite.

Fix any convergent sequence $(x_n)_n$. We may assume that it converges to 0. Suppose first that $(x_n)_n$ contains an infinite decreasing subsequence. In this case it suffices to consider a decreasing $(x_{k_n})_n \subseteq (x_n)_n$ such that:

$$\left|\left\{n \in \omega :\ x_{k_n} \ge \frac{1}{q_i}\right\}\right| < \frac{m}{2},$$

where $k_m \le i < k_{m+1}$. If $(x_n)_n$ does not contain an infinite decreasing subsequence, pick an increasing $(x_{k_n})_n \subseteq (x_n)_n$ such that:

$$\left|\left\{n \in \omega :\ x_{k_n} \le -\frac{1}{q_i}\right\}\right| < \frac{m}{2}.$$

In both cases, let $(y_j^n)_j$ be such that $|x_{k_n}| = f((y_j^n)_j)$. Then

$$\left\{\sum_{j\le i} \frac{y_j^n}{q_j} :\ n \in \omega\right\} = \left\{\sum_{j\le i} \frac{y_j^n}{q_j} :\ n < \frac{m}{2}\right\}$$

and we can proceed in the same way as in the proof of Theorem 5. □

We end with two open questions. We believe that both require developing new methods (besides the ones used in this paper).

Question 3. Is the family of all Haar-countable sets an ideal?

Question 4. Is it true that a countable union of Haar-finite sets must be Haar-countable?

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