IDEAL WEAK QN-SPACES

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Abstract. This paper is devoted to studies of $\mathcal{I}$wQN-spaces and some of their cardinal characteristics.

Recently, Šupina in [33] proved that $\mathcal{I}$ is not a weak P-ideal if and only if any topological space is an $\mathcal{I}$QN-space. Moreover, under $\mathfrak{p}=\mathfrak{c}$ he constructed a maximal ideal $\mathcal{I}$ (which is not a weak P-ideal) for which the notions of $\mathcal{I}$QN-space and QN-space do not coincide. In this paper we show that, consistently, there is an ideal $\mathcal{I}$ (which is not a weak P-ideal) for which the notions of $\mathcal{I}$wQN-space and wQN-space do not coincide. This is a partial solution to [6, Problem 3.7]. We also prove that for this ideal the ideal version of Scheepers Conjecture does not hold (this is the first known example of such weak P-ideal).

We obtain a strictly combinatorial characterization of $\mathtt{non}(\mathcal{I}\text{wQN-space})$ similar to the one given in [33] by Šupina in the case of $\mathtt{non}(\mathcal{I}\text{QN-space})$. We calculate $\mathtt{non}(\mathcal{I}\text{QN-space})$ and $\mathtt{non}(\mathcal{I}\text{wQN-space})$ for some weak P-ideals. Namely, we show that $\mathfrak{b}\leq\mathtt{non}(\mathcal{I}\text{QN-space})\leq\mathtt{non}(\mathcal{I}\text{wQN-space})\leq\mathfrak{d}$ for every weak P-ideal $\mathcal{I}$ and that $\mathtt{non}(\mathcal{I}\text{QN-space})=\mathtt{non}(\mathcal{I}\text{wQN-space})=\mathfrak{b}$ for every $\mathtt{F}_\sigma$ ideal $\mathcal{I}$ as well as for every analytic P-ideal $\mathcal{I}$ generated by an unbounded submeasure (this establishes some new bounds for $\mathfrak{b}(\mathcal{I},\mathcal{I},\text{Fin})$ introduced in [32]). As a consequence, we obtain some bounds for $\mathtt{add}(\mathcal{I}\text{QN-space})$. In particular, we get $\mathtt{add}(\mathcal{I}\text{QN-space})=\mathfrak{b}$ for analytic P-ideals $\mathcal{I}$ generated by unbounded submeasures.

By a result of Bukovský, Das and Šupina from [6] it is known that in the case of tall ideals $\mathcal{I}$ the notions of $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space) and QN-space (wQN-space) cannot be distinguished. Answering [6, Problem 3.2], we prove that if $\mathcal{I}$ is a tall ideal and X is a topological space of cardinality less than $\mathtt{cov}^*(\mathcal{I})$, then X is an $\mathcal{I}$wQN-space if and only if it is a wQN-space.

1. Preliminaries

The paper is organized as follows. In this Section we introduce basic notions which will be used in further considerations. Section 2 is devoted to our main results. We investigate $\mathtt{non}(\mathcal{I}\text{QN-space})$ and $\mathtt{non}(\mathcal{I}\text{wQN-space})$ for weak P-ideals. Among other, we show that $\mathtt{non}(\mathcal{I}\text{QN-space})=\mathtt{non}(\mathcal{I}\text{wQN-space})=\mathfrak{b}$ for every $\mathtt{F}_\sigma$ ideal $\mathcal{I}$ as well as for every analytic P-ideal $\mathcal{I}$. We also prove that, consistently, there is an ideal $\mathcal{I}$ (which is not a weak P-ideal) for which the notions of $\mathcal{I}$wQN-space and wQN-space do not coincide. This leads us to a conclusion that the ideal version of Scheepers Conjecture does not hold even for some weak P-ideals. In Section 3 we show that for any tall ideal $\mathcal{I}$ and a topological space X of cardinality less than $\mathtt{cov}^*(\mathcal{I})$ the notions of $\mathcal{I}$wQN-space and wQN-space coincide (this solves [6, Problem 3.2]). Section 4 is devoted to some remarks concerning additivity of $\mathcal{I}$QN-spaces.

Key words and phrases. QN-spaces; wQN-spaces; Ideal; Ideal convergence; Quasi-normal convergence; Equal convergence; Bounding number; P-ideal.

1.1. **Ideals.** A collection $\mathcal{I}$ of subsets of some set M is called an *ideal on* M if it is closed under taking finite unions and subsets, contains all finite subsets of M and is a proper subset of $\mathcal{P}(M)$. In this paper we consider only ideals on countable sets. In the theory of ideals a special role is played by the ideal $\mathrm{Fin} = [\omega]^{<\omega}$.

Ideals $\mathcal{I}$ (on a set M) and $\mathcal{J}$ (on a set N) are *isomorphic* if there is a bijection $f : N \to M$ such that:

$$A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}.$$

Results of this paper, although formulated only for ideals on ω, can be generalized for ideals on arbitrary countable sets with the use of isomorphisms.

In our further considerations we will also need the following order on ideals. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω. We say that $\mathcal{I}$ *is below* $\mathcal{J}$ *in the Katětov order* and write $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}$ for all $A \subseteq \omega$. If f is a finite-to-one function (i.e., $f^{-1}[\{n\}]$ is finite for all $n \in \omega$), then we say that $\mathcal{I}$ *is below* $\mathcal{J}$ *in the Katětov-Blass order* and write $\mathcal{I} \leq_{KB} \mathcal{J}$.

A property of ideals can often be expressed by finding a critical ideal (in sense of some order on ideals) with respect to this property (see [22, Theorem 1.3], [23, Theorem 2] or [31, Theorems 2.1 and 3.3]). This approach is very effective, especially in the context of ideal convergence (see [25] or [26]). One such result, regarding the topic of this paper, will be presented in Subsection 1.4.

An ideal on a set M is called:

- *tall* if every infinite subset of M contains an infinite member of the ideal;
- *maximal* if it is maximal with respect to inclusion of ideals on M, i.e., there is no other (besides $\mathcal{I}$) ideal on M containing $\mathcal{I}$;
- a *P-ideal* if for every $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ of M there is $A \in \mathcal{I}$ with $A_n \setminus A$ finite for all $n \in \omega$;
- a *weak P-ideal* if for every partition $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ of M there is $A \notin \mathcal{I}$ with $A_n \cap A$ finite for all $n \in \omega$.

Clearly, every P-ideal is a weak P-ideal.

$\mathrm{Fin} \otimes \mathrm{Fin}$ is the ideal on $\omega \times \omega$ consisting of all sets $A \subseteq \omega \times \omega$ such that

$$\{n \in \omega : A \cap (\{n\} \times \omega) \text{ is infinite}\} \in \mathrm{Fin}.$$

The fact that $\mathcal{I}$ is a weak P-ideal can be expressed equivalently by $\mathrm{Fin}\otimes\mathrm{Fin} \not\leq_{KB} \mathcal{I}$ (for this and other equivalent definitions of this notion, including the ones using different orders on ideals, see [33, Theorem 3.2]).

If $\mathcal{I}$ is an ideal on ω, then we define the ideal $\mathcal{I} \otimes \emptyset$ on $\omega \times \omega$, consisting of all $A \subseteq \omega \times \omega$ such that:

$$\{n \in \omega : A \cap (\{n\} \times \omega) \neq \emptyset\} \in \mathcal{I}.$$

1.2. **Submeasures on ω.** The space 2^X of all functions $f : X \to 2$ is equipped with the product topology (each space $2 = \{0, 1\}$ carries the discrete topology). We treat $\mathcal{P}(X)$ as the space 2^X by identifying subsets of X with their characteristic functions. All topological and descriptive notions in the context of ideals on X will refer to this topology.

A map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω if $\phi(\emptyset) = 0$ and

$$\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B) \text{ for all } A, B \subseteq \omega.$$

It is *lower semicontinuous* if additionally $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0,\dots,n\})$ for all $A \subseteq \omega$. For a lower semicontinuous submeasure ϕ on ω we define the collections:

$$\mathrm{Fin}(\phi) = \{A \subseteq \omega : \ \phi(A) \text{ is finite}\},$$

$$\mathrm{Exh}(\phi) = \left\{A \subseteq \omega : \ \lim_{n\to\infty} \phi(A \cap \{n, n+1, \dots\}) = 0\right\}.$$

For any lower semicontinuous submeasure ϕ on ω the collection $\mathrm{Exh}(\phi)$ is an $\mathtt{F}_{\sigma\delta}$ P-ideal, while $\mathrm{Fin}(\phi)$ is an $\mathtt{F}_\sigma$ ideal containing $\mathrm{Exh}(\phi)$, provided that ϕ is unbounded (see [14, Lemma 1.2.2]). Mazur proved in [27] that every $\mathtt{F}_\sigma$ ideal is equal to $\mathrm{Fin}(\phi)$ for some lower semicontinuous submeasure ϕ, while in [31] Solecki showed that every analytic P-ideal is equal to $\mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ (see also [14, Theorem 1.2.5]).

We will be particularly interested in analytic P-ideals generated by unbounded submeasures, i.e., such analytic P-ideals $\mathcal{I}$ that there exists a lower semicontinuous submeasure ϕ with $\mathcal{I} = \mathrm{Exh}(\phi)$ and $\phi(\omega) = \infty$. This class contains all $\mathtt{F}_\sigma$ P-ideals, as every such ideal is equal to $\mathrm{Fin}(\phi) = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ (see [14, Theorem 1.2.5]). Good examples of $\mathtt{F}_\sigma$ P-ideals are *summable ideals*, i.e., ideals of the form $\mathcal{I}_f = \{A \subseteq \omega : \ \sum_{n\in A} f(n) < \infty\}$ for $f\colon \omega \to \mathbb{R}_+$ such that $\sum_{n\in\omega} f(n) = \infty$ (cf. [14, Example 1.2.3(c)]). It is easy to see that a summable ideal $\mathcal{I}_f$ is tall if and only if $(f(n))$ converges to 0. There are also analytic P-ideals generated by unbounded submeasures, which are not $\mathtt{F}_\sigma$. A good example is the class of tall density ideals in the sense of Farah, which are not Erdős-Ulam ideals (i.e., the class ($\mathcal{Z}4$) from [14, Lemma 1.13.9]). The class of tall density ideals contains all *simple density ideals*, i.e., ideals of the form $\mathcal{Z}_g = \{A \subseteq \omega : \ \lim_{n\to\infty} \frac{|A\cap n|}{g(n)} = 0\}$ for $g\colon \omega \to (0,\infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\left(\frac{n}{g(n)}\right)$ does not converge to 0 (see [1, Section 3] for details). By [23, Proposition 1], a simple density ideal $\mathcal{Z}_g$ is not an Erdős-Ulam ideal if and only if the sequence $\left(\frac{n}{g(n)}\right)$ is unbounded (equivalently: $\mathcal{Z}_g$ does not contain the classical ideal $\mathcal{Z}_{\mathrm{id}}$ of sets of asymptotic density zero – cf. [23, Theorem 2]). In [24] it is shown that there are $\mathfrak{c}$ many non-isomorphic simple density ideals which are not Erdős-Ulam ideals.

1.3. **Ideal convergence.** Let $\mathcal{I}$ be an ideal on ω. A sequence of reals (x_n) is $\mathcal{I}$-*convergent* to $x \in \mathbb{R}$ if $\{n \in \omega : \ |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for any $\varepsilon > 0$. In this case we write $x_n \xrightarrow{\mathcal{I}} x$. Suppose now that X is a set, $(f_n) \subseteq \mathbb{R}^X$ and $f \in \mathbb{R}^X$. We say that (f_n) is $\mathcal{I}$-*quasi-normally convergent* to f ($f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$) if there is a sequence (ε_n) of positive reals with $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that $\{n \in \omega : \ |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for each $x \in X$. Note that actually in this definition we can require that $(\varepsilon_n) \subseteq (0,1)$.

The above notion generalizes its classical counterpart – Fin-quasi-normal convergence is called *quasi-normal convergence* or *equal convergence* and has been introduced independently by Bukovská in [4] and by Császár and Laczkovich in [10]. In [10] it was shown that quasi-normal convergence is equivalent to σ-uniform convergence.

The ideal version of quasi-normal convergence has been intensively studied e.g. in [11], [12], [16], [17], [26] and [32].

The next proposition shows that if $\mathcal{I} \neq \mathcal{J}$, then $\mathcal{I}$QN-convergence differs from $\mathcal{J}$QN-convergence. In particular, $\mathcal{I}$QN-convergence differs from QN-convergence for all ideals $\mathcal{I} \neq \mathrm{Fin}$ (this solves the first part of [6, Problem 3.7]).

Proposition 1.1. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals with $\mathcal{I}\setminus\mathcal{J}\neq\emptyset$, then for any nonempty set X there is a sequence of real-valued functions defined on X, which $\mathcal{I}QN$-converges to 0 but does not $\mathcal{J}QN$-converge to 0.*

Proof. Let $A\in\mathcal{I}\setminus\mathcal{J}$. Define a sequence $(f_n)\subseteq\mathbb{R}^X$ by:

$$f_n(x)=\begin{cases}1 & \text{if } n\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Then for any sequence $(\varepsilon_n)\subseteq(0,1)$ we have:

$$\{n\in\omega:\ |f_n(x)|\geq\varepsilon_n\}=A\in\mathcal{I}\setminus\mathcal{J}$$

for every $x\in X$. Hence, $f_n\xrightarrow{\mathcal{I}QN}0$ but (f_n) does not $\mathcal{J}$QN-converge to 0. □

1.4. **$\mathcal{I}$QN-spaces and $\mathcal{I}$wQN-spaces.** For any $B\in[\omega]^\omega$ by $(e_B(n))$ we denote its increasing enumeration, i.e., $e_B:\omega\to B$ is the unique bijection satisfying $e_B(n)<e_B(n+1)$.

Let $\mathcal{I}$ be an ideal on ω. A topological space X is called:

- a *QN-space* if any sequence $(f_n)\subseteq\mathbb{R}^X$ of continuous functions converging to zero quasi-normally converges to zero;
- a *wQN-space* if for any sequence $(f_n)\subseteq\mathbb{R}^X$ of continuous functions converging to zero there is a subsequence (f_{n_k}) quasi-normally converging to zero, i.e., there is an infinite $B\subseteq\omega$ such that $(f_{e_B(n)})$ quasi-normally converges to zero;
- an *$\mathcal{I}QN$-space* if any sequence $(f_n)\subseteq\mathbb{R}^X$ of continuous functions converging to zero $\mathcal{I}$-quasi-normally converges to zero;
- an *$\mathcal{I}wQN$-space* if for any sequence $(f_n)\subseteq\mathbb{R}^X$ of continuous functions converging to zero there is a subsequence (f_{n_k}) $\mathcal{I}$-quasi-normally converging to zero, i.e., there is an infinite $B\subseteq\omega$ such that $(f_{e_B(n)})$ $\mathcal{I}$-quasi-normally converges to zero.

QN-spaces and wQN-spaces were introduced by Bukovský, Recław and Repický in [7] while their ideal counterparts were defined by Das and Chandra in [11]. Note that Bukovský, Das and Šupina in [6] use a slightly different definition of an $\mathcal{I}$wQN-space – they allow the sequence (n_k) to be arbitrary, not necessarily increasing. Each $\mathcal{I}$wQN-space (in our sense) fulfills their definition. For more about QN-spaces and wQN-spaces see e.g. [5], [7] or [34]. $\mathcal{I}$QN-spaces and $\mathcal{I}$wQN-spaces are examined e.g. in [11], [6] or [33].

The following diagram presents relations between above notions.

$$\begin{array}{ccc}\text{QN-space} & \longrightarrow & \text{wQN-space}\\ \downarrow & & \downarrow\\ \mathcal{I}\text{QN-space} & \longrightarrow & \mathcal{I}\text{wQN-space}\end{array}$$

Moreover, we have some partial results showing interactions between ideal QN-spaces for different ideals. It is easy to observe that if $\mathcal{I}$ and $\mathcal{J}$ are two ideals on ω such that $\mathcal{I}\subseteq\mathcal{J}$, then any $\mathcal{I}$QN-space is a $\mathcal{J}$QN-space.

Theorem 1.2. *(Šupina,* [33, Proposition 4.3]*) Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω such that $\mathcal{I}\leq_{KB}\mathcal{J}$. Then any $\mathcal{I}QN$-space is a $\mathcal{J}QN$-space.*

Theorem 1.3. *(Bukovský, Das and Šupina,* [6, Corollary 3.4]*) For a non-tall ideal $\mathcal{I}$ on ω the notions of $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space) and QN-space (wQN-space) coincide.*

The above result tells us that non-tall ideals are not interesting in the context of ideal QN-spaces and ideal wQN-spaces. Below we present a result showing that this is the case also for ideals which are not weak P-ideals.

Theorem 1.4. *(Šupina,* [33, Theorem 1.4]*) The following are equivalent for any ideal $\mathcal{I}$ on ω:*

(a) *$\mathcal{I}$ is not a weak P-ideal;*
(b) *every topological space is an $\mathcal{I}$QN-space.*

Observe that the above result implies that for non-weak P-ideals $\mathcal{I}$ any topological space is also an $\mathcal{I}$wQN-space.

1.5. **Some cardinal invariants.** Recall the definition of the *pseudointersection number*:

$$\mathfrak{p} = \min\left\{|\mathcal{A}|:\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall_{\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}} \bigcap \mathcal{A}_0 \neq \emptyset\ \wedge\ \forall_{S \in [\omega]^\omega} \exists_{A \in \mathcal{A}}\ |S \setminus A| = \omega\right\}.$$

Šupina proved that, consistently, the notions of $\mathcal{I}$QN-space and QN-space can be distinguished even for weak P-ideals: if $\mathfrak{p} = \mathfrak{c}$, then there are a maximal ideal $\mathcal{I}$ which is a weak P-ideal and an $\mathcal{I}$QN-space of cardinality $\mathfrak{c}$ which is not a QN-space ([33, Theorem 1.5]). However, the space in this example is a wQN-space. One of the motivations of this paper is to distinguish the notions of wQN-space and $\mathcal{I}$wQN-space in the case of weak P-ideals. This is done in Theorem 2.11.

In our further considerations we will also need the following notions. Let $\mathcal{I}$ be an ideal on ω. If $f, g \in \omega^\omega$, then we write $f \leq_{\mathcal{I}} g$ if $\{n \in \omega:\ f(n) > g(n)\} \in \mathcal{I}$. The cardinals $\mathfrak{b}_{\mathcal{I}}$ and $\mathfrak{d}_{\mathcal{I}}$ denote the minimal cardinalities of an unbounded and dominating family in ω^ω ordered by $\leq_{\mathcal{I}}$. We write $\mathfrak{b}_{\mathrm{Fin}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathrm{Fin}} = \mathfrak{d}$ for convenience. In it easy to observe that $\mathfrak{b} \leq \mathfrak{b}_{\mathcal{I}} \leq \mathfrak{d}_{\mathcal{I}} \leq \mathfrak{d}$ for any ideal $\mathcal{I}$ and that $\mathfrak{b}_{\mathcal{I}} = \mathfrak{d}_{\mathcal{I}}$ for any maximal ideal $\mathcal{I}$.

Let $\mathcal{I}$ be a weak P-ideal on ω. Then:

- $\mathtt{non}$($\mathcal{I}$QN-space) denotes the minimal cardinality of a perfectly normal topological space which is not an $\mathcal{I}$QN-space;
- $\mathtt{non}$($\mathcal{I}$wQN-space) denotes the minimal cardinality of a perfectly normal topological space which is not an $\mathcal{I}$wQN-space.

In the case of $\mathcal{I} = \mathrm{Fin}$, by a result of Bukovský, Recław and Repický we know the exact values: $\mathtt{non}$(QN-space) $=$ $\mathtt{non}$(wQN-space) $= \mathfrak{b}$ (cf. [7, Corollary 3.2]). In [33, Corollary 6.5] it is shown that $\mathtt{non}$($\mathcal{I}$QN-space) has a strictly combinatorial characterization. In Theorem 2.7 we obtain a similar characterization in the case of $\mathtt{non}$($\mathcal{I}$wQN-space).

2. Uniformity of $\mathcal{I}$QN-spaces and $\mathcal{I}$wQN-spaces

Definition 2.1. *For a weak P-ideal $\mathcal{I}$ on ω let $\kappa(\mathcal{I})$ denote the minimal cardinality of a family $\mathcal{A} \subseteq \mathrm{Fin}^\omega$ with the property that for every partition $(B_n)_{n \in \omega \cup \{-1\}}$ of ω satisfying $e_B^{-1}[B_n] \in \mathcal{I}$ for all $n \in \omega$, where $B = \bigcup_{n \in \omega} B_n$, there is $(A_n) \in \mathcal{A}$ such that*

$$e_B^{-1}\left[\bigcup_{n \in \omega} A_n \cap B_n\right] \notin \mathcal{I}.$$

Remark 2.2. *Notice that $\kappa(\mathcal{I})$ can be defined in a slightly different (and perhaps less complicated) way. For an ideal $\mathcal{I}$ on ω denote by $\mathcal{P}_{\mathcal{I}}$ the family of all partitions of ω into sets belonging to $\mathcal{I}$. Then*

$$\kappa(\mathcal{I}) = \min\left\{|\mathcal{A}| : \ \mathcal{A} \subseteq Fin^{\omega} \ \wedge \ \forall_{B\in[\omega]^{\omega}} \ \forall_{(D_n)\in\mathcal{P}_{\mathcal{I}}} \ \exists_{(A_n)\in\mathcal{A}} \bigcup_{n\in\omega} e_B^{-1}[A_n] \cap D_n \notin \mathcal{I}\right\}.$$

Indeed, given $(B_n)_{n\in\omega\cup\{-1\}}$ such as above, it suffices to put $B = \bigcup_{n\in\omega} B_n$ and $D_n = e_B^{-1}[B_n]$ for each $n \in \omega$. On the other hand, given $B \in [\omega]^{\omega}$ and $(D_n) \in \mathcal{P}_{\mathcal{I}}$, put $B_{-1} = \omega \setminus B$ and $B_n = e_B[D_n]$ for each $n \in \omega$ to obtain the required partition.

Theorem 2.3. *Let $\mathcal{I}$ be a weak P-ideal on ω. The following are equivalent for any set X:*

(a) *$|X| < \kappa(\mathcal{I})$;*
(b) *for any sequence of real-valued functions defined on X which converges to some $f \in \mathbb{R}^X$ one can find its subsequence $\mathcal{I}QN$-converging to f;*
(c) *X with the discrete topology is an $\mathcal{I}wQN$-space.*

Proof. This proof is only a slight modification of the proof of [17, Theorem 5.1].

The implication (b)⇒(c) is obvious. We will prove (a)⇒(b) and (c)⇒(a).

(a)⇒(b): Suppose that $|X| < \kappa(\mathcal{I})$ and $(f_n) \subseteq \mathbb{R}^X$ converges to some $f \in \mathbb{R}^X$. Define

$$A_k^x = \left\{n \in \omega : \ |f_n(x) - f(x)| \geq \frac{1}{k+1}\right\} \in \mathrm{Fin}$$

for each $x \in X$ and $k \in \omega$. Let $(B_n)_{n\in\omega\cup\{-1\}}$ be the partition of ω which exists by the definition of $\kappa(\mathcal{I})$ and denote $B = \bigcup_{n\in\omega} B_n$. Define an $\mathcal{I}$-converging to 0 sequence $(\varepsilon_k) \subseteq (0, 1]$ by:

$$\varepsilon_k = \frac{1}{n+1} \iff k \in e_B^{-1}[B_n] \in \mathcal{I}.$$

Fix any $x \in X$ and observe that:

$$\left\{k \in \omega : \ |f_{e_B(k)}(x) - f(x)| \geq \varepsilon_k\right\} = \bigcup_{n\in\omega} \left\{k \in e_B^{-1}[B_n] : \ |f_{e_B(k)}(x) - f(x)| \geq \varepsilon_k\right\} =$$

$$= \bigcup_{n\in\omega} \left\{k \in e_B^{-1}[B_n] : \ |f_{e_B(k)}(x) - f(x)| \geq \frac{1}{n+1}\right\} = e_B^{-1}\left[\bigcup_{n\in\omega} A_n^x \cap B_n\right] \in \mathcal{I}.$$

(c)⇒(a): Suppose to the contrary that $|X| \geq \kappa(\mathcal{I})$. Let $\phi\colon \kappa(\mathcal{I}) \to X$ be an injection. Suppose that $\mathcal{A} = \{(A_n^{\alpha}) : \ \alpha < \kappa(\mathcal{I})\}$ is the family from the definition of $\kappa(\mathcal{I})$. Define on X real-valued functions (f_n) by:

$$f_n(x) = \begin{cases} \frac{1}{k+1} & \text{if } n \in A_k^{\alpha} \setminus \bigcup_{m<k} A_m^{\alpha} \text{ and } x = \phi(\alpha) \text{ for some } \alpha < \kappa(\mathcal{I}), \\ 0 & \text{otherwise,} \end{cases}$$

for each $n \in \omega$. Then f_n converges to 0, so by our assumption it has a subsequence (f_{n_k}) which $\mathcal{I}$QN-converges to 0. Let $(\varepsilon_k) \subseteq (0, 1)$ be the $\mathcal{I}$-converging to 0 sequence witnessing it.

Define:

$$B = \{n_k : \ k \in \omega\}, \quad B_{-1} = \omega \setminus B \quad \text{and}$$

$$B_n = \left\{n_k : \ \frac{1}{n+2} \leq \varepsilon_k < \frac{1}{n+1}\right\} \quad \text{for all} \quad n \in \omega$$

(observe that $e_B(k) = n_k$ for each $k \in \omega$). Then $(B_n)_{n\in\omega\cup\{-1\}}$ is a partition of ω and $e_B^{-1}[B_n] = \{k \in \omega : \ n_k \in B_n\} \in \mathcal{I}$ for all $n \in \omega$ as (ε_k) is $\mathcal{I}$-converging to 0. Therefore, there is $\alpha_0 < \kappa(\mathcal{I})$ such that:

$$e_B^{-1}\left[\bigcup_{n\in\omega} A_n^{\alpha_0} \cap B_n\right] \notin \mathcal{I}.$$

Denote the above set by C_{α_0}. We will show that:

$$C_{\alpha_0} \subseteq \{k \in \omega : \ |f_{n_k}(\phi(\alpha_0))| \geq \varepsilon_k\},$$

which will contradict $f_{n_k} \xrightarrow{\mathcal{I}\text{QN}} 0$. Fix $k \in C_{\alpha_0}$. Then there is $i \in \omega$ such that $n_k \in B_i$ and $n_k \in A_i^{\alpha_0}$. Hence,

$$|f_{n_k}(\phi(\alpha_0))| \geq \frac{1}{i+1} > \varepsilon_k.$$

□

Lemma 2.4. *$\kappa(\mathcal{I}) \leq \mathfrak{d}$ for every weak P-ideal $\mathcal{I}$ on ω.*

Proof. Let $\mathcal{F} = \{f_\alpha \in \omega^\omega : \ \alpha < \mathfrak{d}\}$ be a dominating family. Define finite sets:

$$A_n^\alpha = \{k \in \omega : \ k \leq f_\alpha(n)\}$$

for each $\alpha < \mathfrak{d}$ and $n \in \omega$. We claim that the family $\{(A_n^\alpha) : \ \alpha < \mathfrak{d}\}$ witnesses $\kappa(\mathcal{I}) \leq \mathfrak{d}$.

Fix any partition $(B_n)_{n\in\omega\cup\{-1\}}$ of ω such that $e_B^{-1}[B_n] \in \mathcal{I}$ for each $n \in \omega$, where B denotes the set $\bigcup_{n\in\omega} B_n$. Since $\mathcal{I}$ is a weak P-ideal, there is $C \notin \mathcal{I}$ with $C \cap e_B^{-1}[B_n]$ finite for all $n \in \omega$. Define a function $g \in \omega^\omega$ by:

$$g(n) = \max\left((e_B[C] \cap B_n) \cup \{0\}\right).$$

Since $\mathcal{F}$ is dominating, there is $\alpha_0 < \mathfrak{d}$ with $g \leq^* f_{\alpha_0}$, i.e., $F = \{n \in \omega : g(n) > f_{\alpha_0}(n)\}$ is finite. As $(e_B^{-1}[B_n])$ is a partition of ω, we have $\mathcal{I} \not\ni C = \bigcup_{n\in\omega} e_B^{-1}[B_n] \cap C$. Moreover, $\bigcup_{n\in\omega\setminus F} e_B^{-1}[B_n] \cap C \notin \mathcal{I}$ since $\bigcup_{n\in F} e_B^{-1}[B_n] \cap C$ is finite. We will show that:

$$\bigcup_{n\in\omega\setminus F} e_B^{-1}[B_n] \cap C \subseteq \bigcup_{n\in\omega} e_B^{-1}\left[B_n \cap A_n^{\alpha_0}\right].$$

Indeed, let $i \in e_B^{-1}[B_n] \cap C$ for some $n \in \omega \setminus F$. Then $e_B(i) \leq g(n) \leq f_{\alpha_0}(n)$. Hence, $e_B(i) \in A_n^{\alpha_0}$ and $i \in e_B^{-1}\left[B_n \cap A_n^{\alpha_0}\right]$. □

Lemma 2.5. *The following are equivalent for any ideal $\mathcal{I}$:*

(a) *$\mathcal{I}$ is a subset of some $\mathtt{F}_\sigma$ ideal;*
(b) *$\mathcal{I}$ is $\leq_{KB}$-below some $\mathtt{F}_\sigma$ ideal;*
(c) *$\mathcal{I}$ is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal.*

Proof. The implications (a)⇒(b) and (b)⇒(c) are obvious. To prove (c)⇒(a) suppose that $\mathcal{I}$ is an ideal on a set M which is $\leq_K$-below $\mathtt{F}_\sigma$ ideal $\mathcal{J}$ on a set N. Let $f\colon N \to M$ be the witnessing function. Then $\bar{f}\colon \mathcal{P}(M) \to \mathcal{P}(N)$ given by $\bar{f}(A) = f^{-1}[A]$ for each $A \in \mathcal{P}(M)$ is continuous. Observe that $\bar{f}^{-1}[\mathcal{J}]$ is an ideal on M. Hence, it is an $\mathtt{F}_\sigma$ ideal, since $\mathcal{J}$ is $\mathtt{F}_\sigma$. Finally, $\mathcal{I} \subseteq \bar{f}^{-1}[\mathcal{J}]$, since f witnesses $\mathcal{I} \leq_K \mathcal{J}$. □

Lemma 2.6. *$\kappa(\mathcal{I}) \leq \mathfrak{b}$ for every ideal $\mathcal{I}$ on ω which is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal.*

Proof. Observe that $\mathcal{I}\subseteq\mathcal{J}$ implies $\kappa(\mathcal{I})\leq\kappa(\mathcal{J})$. Therefore, by Lemma 2.5, we only need to show that $\kappa(\mathcal{J})\leq\mathfrak{b}$ for every $\mathtt{F}_\sigma$ ideal $\mathcal{J}$.

Let $\mathcal{F}=\{f_\alpha\in\omega^\omega:\ \alpha<\mathfrak{b}\}$ be an unbounded family. Without loss of generality we can assume that each f_α is non-decreasing (we may replace f_α with $\hat{f}_\alpha$ given by $\hat{f}_\alpha(n)=\max_{i\leq n}f(i)$ and observe that the family $\{\hat{f}_\alpha\in\omega^\omega:\ \alpha<\mathfrak{b}\}$ is unbounded as $f_\alpha\leq\hat{f}_\alpha$ for each α). Define finite sets:

$$A_n^\alpha=\{k\in\omega:\ k\leq f_\alpha(n)\}$$

for each $\alpha<\mathfrak{b}$ and $n\in\omega$. We claim that the family $\{(A_n^\alpha):\ \alpha<\mathfrak{b}\}$ witnesses $\kappa(\mathcal{J})\leq\mathfrak{b}$.

Let ϕ be the lower semi-continuous submeasure such that $\mathcal{J}=\mathrm{Fin}(\phi)$. Fix any partition $(B_n)_{n\in\omega\cup\{-1\}}$ of ω such that $e_B^{-1}[B_n]\in\mathcal{J}$ for each $n\in\omega$, where B denotes the set $\bigcup_{n\in\omega}B_n$.

Define a function $g\in\omega^\omega$ by $g(0)=0$ and

$$g(n)=\min\left\{k\in\omega:\ \phi\left(e_B^{-1}\left[[0,k]\right]\right)-\phi\left(e_B^{-1}\left[\bigcup_{i<n}B_i\right]\right)\geq n\right\}$$

for $n>0$. Note that g is well-defined. Indeed, it follows from the facts that $\phi\left(e_B^{-1}\left[\bigcup_{i<n}B_i\right]\right)$ is finite and $\phi\left(e_B^{-1}\left[[0,k]\right]\right)$ tends to infinity as $k\to\infty$.

Recall that $\mathcal{F}$ is unbounded. Hence, there is $\alpha_0<\mathfrak{b}$ with $g(n)\leq f_{\alpha_0}(n)$ for infinitely many n. For each such n we have:

$$\phi\left(e_B^{-1}\left[\bigcup_{i\geq n}B_i\cap A_n^{\alpha_0}\right]\right)\geq\phi\left(e_B^{-1}\left[A_n^{\alpha_0}\right]\right)-\phi\left(e_B^{-1}\left[\bigcup_{i<n}B_i\right]\right)\geq$$

$$\geq\phi\left(e_B^{-1}\left[[0,g(n)]\right]\right)-\phi\left(e_B^{-1}\left[\bigcup_{i<n}B_i\right]\right)\geq n.$$

Now it suffices to observe that:

$$\bigcup_{n\in\omega}e_B^{-1}\left[B_n\cap A_n^{\alpha_0}\right]\supseteq\bigcup_{n\in\omega}e_B^{-1}\left[\bigcup_{i\geq n}B_i\cap A_n^{\alpha_0}\right]\notin\mathcal{J}.$$

Indeed, if $k\in\bigcup_{n\in\omega}e_B^{-1}\left[\bigcup_{i\geq n}B_i\cap A_n^{\alpha_0}\right]$, then there are $n\in\omega$ and $i\geq n$ with $e_B(k)\in B_i\cap A_n^{\alpha_0}$. However, as f_{α_0} is non-decreasing, $A_n^{\alpha_0}\subseteq A_i^{\alpha_0}$. Therefore, $e_B(k)\in B_i\cap A_i^{\alpha_0}$. $\square$

Theorem 2.7. *We have:*

(a) $\mathfrak{b}\leq\mathtt{non}(\mathcal{I}$*QN-space*$)\leq\mathtt{non}(\mathcal{I}$*wQN-space*$)=\kappa(\mathcal{I})\leq\mathfrak{d}$ *for every weak P-ideal* $\mathcal{I}$ *on* ω*;*

(b) $\mathtt{non}(\mathcal{I}$*QN-space*$)=\mathtt{non}(\mathcal{I}$*wQN-space*$)=\kappa(\mathcal{I})=\mathfrak{b}$ *for every ideal* $\mathcal{I}$ *on* ω *which is* $\leq_K$*-below some* $\mathtt{F}_\sigma$ *ideal.*

Proof. The equality $\mathtt{non}(\mathcal{I}\text{wQN-space})=\kappa(\mathcal{I})$ follows immediately from Theorem 2.3 (when showing that $\mathtt{non}(\mathcal{I}\text{wQN-space})\leq\kappa(\mathcal{I})$, it suffices to endow X with the discrete topology).

(a): The first inequality follows from $\mathfrak{b}=\mathtt{non}(\text{QN-space})$ and the fact that every QN-space is an $\mathcal{I}$QN-space for any ideal $\mathcal{I}$. The second inequality is obvious, as every $\mathcal{I}$QN-space is an $\mathcal{I}$wQN-space. The third one is shown above and the last one is Lemma 2.4.

(b): We have

$$\mathfrak{b} = \mathtt{non}(\text{QN-space}) \leq \mathtt{non}(\mathcal{I}\text{QN-space}) \leq \mathtt{non}(\mathcal{I}\text{wQN-space}) = \kappa(\mathcal{I}).$$

Moreover, $\kappa(\mathcal{I}) \leq \mathfrak{b}$ by Lemma 2.6. □

Corollary 2.8. *$\mathtt{non}(\mathcal{I}QN\text{-space}) = \mathtt{non}(\mathcal{I}wQN\text{-space}) = \mathfrak{b}$ for every analytic P-ideal $\mathcal{I}$ on ω generated by an unbounded submeasure.*

Proof. Any analytic P-ideal is of the form $\text{Exh}(\phi)$ for some lower semi-continuous submeasure ϕ. Therefore, it is contained in (so, in particular, $\leq_K$-below) $\text{Fin}(\phi)$, which is $\mathtt{F}_\sigma$ (cf. [14]). If ϕ is unbounded, then $\omega \notin \text{Fin}(\phi)$, so $\text{Fin}(\phi)$ becomes an ideal and we are done. □

Remark 2.9. *Note that $\mathtt{non}(\mathcal{I}QN\text{-space}) = \mathtt{non}(\mathcal{I}wQN\text{-space}) = \mathfrak{b}$ also for some non-analytic ideals. Indeed, let $\mathcal{I}$ be a non-analytic ideal and consider the ideal $Fin \oplus \mathcal{I}$ on $\{0,1\} \times \omega$ given by:*

$$A \in Fin \oplus \mathcal{I} \iff \{n \in \omega : \ (0,n) \in A\} \in Fin \ \wedge \ \{n \in \omega : \ (1,n) \in A\} \in \mathcal{I}$$

for every $A \subseteq \{0,1\} \times \omega$. Then $Fin \oplus \mathcal{I}$ is not analytic and non-tall (so, by Theorem 1.3, a topological space is a QN-space if and only if it is an $(Fin \oplus \mathcal{I})$QN-space).

Corollary 2.10. *We have the following:*

(a) *$\mathfrak{b}(\mathcal{I}, \mathcal{I}, Fin) = \mathfrak{b}$ for every ideal $\mathcal{I}$ on ω which is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal. In particular, $\mathfrak{b}(\mathcal{I}, \mathcal{I}, Fin) = \mathfrak{b}$ for all $\mathtt{F}_\sigma$ ideals and all analytic P-ideals generated by unbounded submeasures.*

(b) *$\mathfrak{b} \leq \mathfrak{b}(\mathcal{I}, \mathcal{I}, Fin) \leq \mathfrak{d}$ for every weak P-ideal $\mathcal{I}$ on ω.*

Proof. By [33, Section 6] and the definition of $\mathfrak{b}(\mathcal{I}, \mathcal{I}, \text{Fin})$ (see [32]), we have $\mathfrak{b}(\mathcal{I}, \mathcal{I}, \text{Fin}) = \mathtt{non}(\mathcal{I}\text{QN-space})$. □

Theorem 2.11. *If $\mathfrak{b} < \mathfrak{b}_\mathcal{J}$ for some ideal $\mathcal{J}$ on ω, then there are a weak P-ideal $\mathcal{I}$ on ω and an $\mathcal{I}$wQN-space which is not a wQN-space.*

Before proving the above, let us make a short comment.

Remark 2.12. *Note that it is consistent with ZFC that $\mathfrak{b} < \mathfrak{b}_\mathcal{J}$ for some ideal $\mathcal{J}$ on ω: in [8] it is proved (in ZFC) that there is a maximal ideal $\mathcal{J}$ with $\mathfrak{b}_\mathcal{J}$ equal to $cf(\mathfrak{d})$ (the cofinality of $\mathfrak{d}$) and consistency of $\mathfrak{b} < cf(\mathfrak{d})$ follows for instance from [2, Theorem 2.5]. Consistency of $\mathfrak{b} < \mathfrak{b}_\mathcal{J}$ may also be obtained under other set-theoretic assumptions – see [29] for details.*

Theorem 2.11 follows from the next Lemma as $\mathtt{non}(\text{wQN}) = \mathfrak{b}$ by [7, Corollary 3.2].

Lemma 2.13. *Let $\mathcal{J}$ be an ideal on ω. Then there is a weak P-ideal $\mathcal{I}$ on ω such that $\mathtt{non}(\mathcal{I}wQN\text{-space}) \geq \mathfrak{b}_\mathcal{J}$.*

Proof. Define an ideal $\bar{\mathcal{I}}$ on $\omega \times \omega$ by:

$$A \in \bar{\mathcal{I}} \iff \{n \in \omega : \ |A \cap (\{n\} \times \omega)| = \omega\} \in \text{Fin} \wedge \{n \in \omega : \ A \cap (\{n\} \times \omega) \neq \emptyset\} \in \mathcal{J}$$

for each $A \subseteq \omega \times \omega$ (i.e., $\bar{\mathcal{I}} = (\text{Fin} \otimes \text{Fin}) \cap (\mathcal{J} \otimes \emptyset)$). Note that $\bar{\mathcal{I}}$ is an ideal as an intersection of two ideals.

We need to show two facts:

(i) $\bar{\mathcal{I}}$ is a weak P-ideal;

(ii) $\mathtt{non}(\mathcal{I}\text{wQN-space}) \geq \mathfrak{b}_{\mathcal{J}}$ for every ideal $\mathcal{I}$ on ω isomorphic to $\bar{\mathcal{I}}$.

Then any ideal on ω isomorphic to $\bar{\mathcal{I}}$ will be as needed (since being a weak P-ideal is invariant over isomorphisms of ideals).

(i) $\bar{\mathcal{I}}$ is a weak P-ideal: Fix a partition (X_n) of $\omega \times \omega$ into sets belonging to $\bar{\mathcal{I}}$. Define by induction two sequences $(m_n), (k_n) \subseteq \omega$ such that for each $n \in \omega$ we have $(n, m_n) \in X_{k_n}$ and

$$m_n = \begin{cases} \min\{m \in \omega : (n,m) \notin \bigcup\{X_{k_i} : i < n\}\} & \text{if } \{n\} \times \omega \not\subseteq \bigcup_{i<n} X_{k_i}, \\ \min\{m \in \omega : (n,m) \in \bigcup\{X_k : |X_k \cap (\{n\} \times \omega)| = \omega\}\} & \text{otherwise.} \end{cases}$$

Then $Y = \{(n, m_n) : \ n \in \omega\} \notin \mathcal{I}$ as $\{n \in \omega : \ Y \cap (\{n\} \times \omega) \neq \emptyset\} = \omega \notin \mathcal{J}$. Moreover, $Y \cap X_n$ is finite for all n (otherwise we would have $|X_n \cap (\{k\} \times \omega)| = \omega$ for infinitely many $k \in \omega$).

(ii) $\mathtt{non}(\mathcal{I}$wQN-space$) \geq \mathfrak{b}_{\mathcal{J}}$: Fix any bijection $\phi\colon \omega \to \omega \times \omega$ and denote $X_n = \phi^{-1}[\{n\} \times \omega]$ for all $n \in \omega$. We will show that $\mathfrak{b}_{\mathcal{J}} \leq \mathtt{non}(\mathcal{I}_\phi\text{wQN-space})$ where $\mathcal{I}_\phi = \{\phi^{-1}[A] : \ A \in \bar{\mathcal{I}}\}$ is an ideal on ω isomorphic to $\bar{\mathcal{I}}$.

We will use the equality $\mathtt{non}(\mathcal{I}\text{wQN-space}) = \kappa(\mathcal{I})$ from Theorem 2.7. Let $\kappa < \mathfrak{b}_{\mathcal{J}}$ and $\{(A^\alpha_n) : \ \alpha < \kappa\} \subseteq \mathrm{Fin}^\omega$. Define:

$$f_\alpha(n) = \max\{k \in \omega : \ (n,k) \in \phi[A^\alpha_n]\}$$

for all $\alpha < \kappa$ and $n \in \omega$. Then there is $g \in \omega^\omega$ such that $\{n \in \omega : \ f_\alpha(n) > g(n)\} \in \mathcal{J}$ for each $\alpha < \kappa$.

Now we proceed to the construction of a partition $(B_n)_{n \in \omega \cup \{-1\}}$. Define:

$$C_{-1} = \{(i,j) : \ j \leq g(i)\};$$
$$C_n = \{(n,j) : \ j > g(n)\}$$

for each $n \in \omega$. Observe that $\phi^{-1}[C_n] \notin \mathrm{Fin}$ and $\phi^{-1}[C_n] \subseteq X_n$ for each n. Define a sequence $(m_k) \in \omega^\omega$ by $k \in X_{m_k}$. Pick inductively a sequence $(n_k) \subseteq \omega$ such that:

$$n_0 = \min\left\{n \in \omega : \ n \in X_{m_0} \cap \phi^{-1}[C_{m_0}]\right\};$$
$$n_k = \min\left\{n > n_{k-1} : \ n \in X_{m_k} \cap \phi^{-1}[C_{m_k}]\right\}.$$

Denote:

$$B = \{n_k : \ k \in \omega\}, \quad B_{-1} = \omega \setminus B \quad \text{and} \quad B_n = B \cap X_n \text{ for each } n \in \omega.$$

Notice that $e_B(k) = n_k$ and for each $n \in \omega$ we have:

(a) $B_n \subseteq X_n$;

(b) $B_n \subseteq \phi^{-1}[C_n]$;

(c) $n_k \in B_{m_k}$.

Moreover, by item (c), we have:

$$k \in e_B^{-1}[B_n] \iff n_k \in B_n \iff n = m_k \iff k \in X_n,$$

which establishes:

(d) $e_B^{-1}[B_n] = X_n \in \mathcal{I}_\phi$.

By item (d), the partition $(B_n)_{n \in \omega \cup \{-1\}}$ will be as needed, provided that we will show $\bigcup_{k \in \omega} e_B^{-1}[A^\alpha_k \cap B_k] \in \mathcal{I}_\phi$ for each $\alpha < \kappa$.

Fix any $\alpha < \kappa$. Then $Y_\alpha = \{n \in \omega : \ f_\alpha(n) > g(n)\} \in \mathcal{J}$. Let $n \in \omega$.

If $n \notin Y_\alpha$, we have:

$$\phi[A^\alpha_n] \cap C_n = \emptyset \implies A^\alpha_n \cap \phi^{-1}[C_n] = \emptyset \implies A^\alpha_n \cap B_n = \emptyset \implies e_B^{-1}[A^\alpha_n \cap B_n] = \emptyset.$$

Indeed, the second implication follows from condition (b) and the remaining two are trivial. By item (d) and the fact that (X_n) is a partition of ω, we get that:

$$\left(\bigcup_{k\in\omega} e_B^{-1}[A_k^\alpha \cap B_k]\right) \cap X_n = \left(\bigcup_{k\in\omega} e_B^{-1}[A_k^\alpha] \cap X_k\right) \cap X_n =$$
$$= e_B^{-1}[A_n^\alpha] \cap X_n = e_B^{-1}[A_n^\alpha \cap B_n] = \emptyset.$$

On the other hand, if $n \in Y_\alpha$, then we have $e_B^{-1}[A_n^\alpha \cap B_n] \in \mathrm{Fin}$ as $A_n^\alpha \in \mathrm{Fin}$. Therefore, again by item (d) and the fact that (X_n) is a partition of ω we get that $\left(\bigcup_{k\in\omega} e_B^{-1}[A_k^\alpha \cap B_k]\right) \cap X_n \in \mathrm{Fin}$.

Hence, $\bigcup_{k\in\omega} e_B^{-1}[A_k^\alpha \cap B_k] \in \mathcal{I}_\phi$. □

Let $\mathcal{I}$ be an ideal on ω. A sequence (U_n) of subsets of a topological space X is an *$\mathcal{I}$-γ-cover* if $U_n \neq X$ for all $n \in \omega$ and $\{n \in \omega :\ x \notin U_n\} \in \mathcal{I}$ for all $x \in X$. By $\mathcal{I}$-Γ we denote the family of all open $\mathcal{I}$-γ-covers. We write Γ instead of Fin-Γ. Moreover, X is $S_1(\Gamma, \mathcal{I}\text{-}\Gamma)$ whenever for every sequence $(\mathcal{U}_n) \subseteq \Gamma$ one can find $U_n \in \mathcal{U}_n$, for $n \in \omega$, with $(U_n) \in \mathcal{I}$-Γ.

The *Scheepers Conjecture* asserts that a space is a wQN-space if and only if it satisfies $S_1(\Gamma, \Gamma)$ (cf. [30]). It is still open whether the Scheepers Conjecture is provable, however Dow showed that it is consistently true (cf. [13]).

Šupina proved in [33, Corollary 1.7] that the ideal version of Scheepers Conjecture does not hold if $\mathcal{I}$ is not a weak P-ideal as in this case one can find a perfectly normal $\mathcal{I}$wQN-space which is not $S_1(\Gamma, \mathcal{I}\text{-}\Gamma)$. However, by Theorem 1.4, if $\mathcal{I}$ is not a weak P-ideal, then any topological space is an $\mathcal{I}$wQN-space, so the above result is not rewarding.

The following result shows that the ideal version of Scheepers Conjecture for weak P-ideals consistently does not hold.

Corollary 2.14. *If $\mathfrak{b} < \mathfrak{b}_\mathcal{J}$ for some ideal $\mathcal{J}$ on ω, then there are a weak P-ideal $\mathcal{I}$ on ω and an $\mathcal{I}$wQN-space which is not $S_1(\Gamma, \mathcal{I}\text{-}\Gamma)$.*

Proof. By [33, Corollary 7.4(ii)], $\mathtt{non}(S_1(\Gamma, \hat{\mathcal{I}}\text{-}\Gamma)\text{-space}) = \mathfrak{b}_{\hat{\mathcal{I}}}$ for every ideal $\hat{\mathcal{I}}$. Let $\mathcal{I}$ be the ideal from Theorem 2.11. We will show that $\mathfrak{b}_\mathcal{I} = \mathfrak{b}$. Observe that if $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\mathfrak{b}_{\mathcal{I}_1} \leq \mathfrak{b}_{\mathcal{I}_2}$. Moreover, $\mathcal{I} \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$. Hence, it suffices to show that $\mathfrak{b}_{\mathrm{Fin}\otimes\mathrm{Fin}} = \mathfrak{b}$. This follows from the fact that $\mathrm{Fin} \otimes \mathrm{Fin}$ is a Borel (in fact $\mathtt{F}_{\sigma\delta\sigma}$) ideal. Indeed, by the proof of [15, Corollary 5.5], we have $\mathfrak{b}_{\hat{\mathcal{I}}} = \mathfrak{b}$ for any ideal $\hat{\mathcal{I}}$ which is $\leq_{RB}$-above Fin and this is the case for every Borel ideal by [14, Corollary 3.10.2]. □

3. Relation between $\mathcal{I}$wQN-spaces and wQN-spaces

In [6, Problem 3.2] authors ask about existence of a tall ideal $\mathcal{I}$ such that for any sequence of functions $\mathcal{I}$QN-converging to 0 one can find its subsequence converging quasi-normally to 0. In this section we investigate this property.

Let $\mathcal{I}$ be a tall ideal on ω. Define:

$$\mathtt{cov}^*(\mathcal{I}) = \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \mathcal{I}\ \wedge\ \forall_{S\in[\omega]^\omega}\ \exists_{A\in\mathcal{A}}\ |A \cap S| = \omega\right\}.$$

This cardinal invariant was considered e.g. in [3] (where a different notation is used) and [18]. It is a variation of the pseudointersection number $\mathfrak{p}$ – we additionally require that the witnessing family is from the filter dual to the ideal $\mathcal{I}$.

Remark 3.1. *We have $\mathfrak{p} \leq \mathtt{cov}^*(\mathcal{I}) \leq \mathfrak{c}$ for every tall ideal $\mathcal{I}$ (cf. [19]).*

There are examples of tall ideals $\mathcal{I}$ with non-trivial values of $\texttt{cov}^*(\mathcal{I})$, for instance:

- $\texttt{cov}^*(\mathrm{Fin}\otimes\mathrm{Fin}) = \mathfrak{b}$ (cf. [19]);
- $\texttt{cov}^*(\texttt{nwd}) = \texttt{cov}(\mathcal{M})$, where $\texttt{nwd}$ is the ideal on $\mathbb{Q}\cap[0,1]$ consisting of all nowhere dense subsets of $\mathbb{Q}\cap[0,1]$ (cf. [19] or [21]);
- $\texttt{cov}^*(\mathcal{ED}) = \texttt{non}(\mathcal{M})$, where $\mathcal{ED}$ is the ideal on $\omega\times\omega$ generated by all vertical lines (i.e., sets $\{n\}\times\omega$ for $n\in\omega$) and graphs of functions from ω to ω (cf. [19] or [20]);
- $\texttt{cov}^*(\texttt{conv}) = \mathfrak{c}$, where $\texttt{conv}$ is the ideal on $\mathbb{Q}\cap[0,1]$ generated by sequences in $\mathbb{Q}\cap[0,1]$ convergent in $[0,1]$ (cf. [19] or [28]).

For more examples see [19].

Theorem 3.2. *Let $\mathcal{I}$ be a tall ideal on ω. The following are equivalent for any set X:*

(a) $|X| < \texttt{cov}^*(\mathcal{I})$*;*
(b) *for any sequence of real-valued functions defined on X, if it $\mathcal{I}$QN-converges to some $f\in\mathbb{R}^X$, then one can find its subsequence QN-converging to f.*

Proof. **(a)⇒(b):** Suppose that $|X| < \texttt{cov}^*(\mathcal{I})$ and fix $(f_n)\subseteq\mathbb{R}^X$ which $\mathcal{I}$QN-converges to some $f\in\mathbb{R}^X$ with the witnessing sequence $(\varepsilon_n)\subseteq(0,1)$.

For each $x\in X$ let:

$$B_x = \{n\in\omega:\ |f_n(x)-f(x)|\geq\varepsilon_n\}\in\mathcal{I}.$$

Let also:

$$A_k = \left\{n\in\omega:\ \frac{1}{k+2}\leq\varepsilon_n<\frac{1}{k+1}\right\}\in\mathcal{I}$$

for all $k\in\omega$.

Observe that $|X|+\omega < \texttt{cov}^*(\mathcal{I})$ as $\texttt{cov}^*(\mathcal{I})\geq\mathfrak{p}>\omega$ (cf. Remark 3.1). Hence, there is an infinite set $S\subseteq\omega$ which has finite intersections with all B_x, $x\in X$, as well as with all A_k, $k\in\omega$. Let (n_m) be an increasing enumeration of S.

We will show that (f_{n_m}) QN-converges to f. Define $\varepsilon'_m=\varepsilon_{n_m}$ for all $m\in\omega$. Then ε'_m converges to 0 as $S\cap A_k$ is finite for each $k\in\omega$. Moreover, we have

$$|\{m\in\omega:\ |f_{n_m}(x)-f(x)|\geq\varepsilon'_m\}| = |S\cap B_x| < \omega$$

for all $x\in X$.

(b)⇒(a): Suppose that $|X|\geq\texttt{cov}^*(\mathcal{I})$. Let $\phi\colon \texttt{cov}^*(\mathcal{I})\to \texttt{X}$ be an injection. Let also $\{A_\alpha:\ \alpha<\texttt{cov}^*(\mathcal{I})\}$ be such a family of members of $\mathcal{I}$ that for each infinite $S\subseteq\omega$ there is $\alpha<\texttt{cov}^*(\mathcal{I})$ with $S\cap A_\alpha$ infinite.

Define a sequence $(f_n)\subseteq\mathbb{R}^X$ by:

$$f_n(x) = \begin{cases} 1 & \text{if } n\in A_\alpha \text{ and } x=\phi(\alpha) \text{ for some } \alpha<\texttt{cov}^*(\mathcal{I}),\\ 0 & \text{otherwise.}\end{cases}$$

Let also $f\in\mathbb{R}^X$ be the function constantly equal to 0.

It is easy to see that for any $(\varepsilon_n)\subseteq(0,1)$ converging to 0 we have:

$$\{n\in\omega:\ |f_n(x)-f(x)|\geq\varepsilon_n\}=\emptyset$$

for each $x\in X\setminus\phi[\texttt{cov}^*(\mathcal{I})]$ and

$$\{n\in\omega:\ |f_n(x)-f(x)|\geq\varepsilon_n\}=A_\alpha\in\mathcal{I}$$

for each $x=\phi(\alpha)\in\phi[\texttt{cov}^*(\mathcal{I})]$. Hence, (f_n) is $\mathcal{I}$QN-convergent to f.

Now we will show that none of subsequences of (f_n) converges to f. Fix any subsequence $(f_{n_m}) \subseteq (f_n)$. The set $S = \{n_m : m \in \omega\}$ is infinite, so there is $\alpha_0 < \mathtt{cov}^*(\mathcal{I})$ with $S \cap A_{\alpha_0}$ infinite. Then we have:

$$|\{m \in \omega : |f_{n_m}(\phi(\alpha_0)) - f(\phi(\alpha_0))| \geq 1\}| = |S \cap A_{\alpha_0}| = \omega.$$

Therefore, (f_{n_m}) cannot QN-converge to f. □

As a consequence of the above Theorem we obtain the main result of this Section.

Corollary 3.3. *Let $\mathcal{I}$ be a tall ideal on ω and X be a topological space of cardinality less than $\mathtt{cov}^*(\mathcal{I})$. Then X is an $\mathcal{I}$wQN-space if and only if it is a wQN-space.*

Proof. Any wQN-space is an $\mathcal{I}$wQN-space and the other implication is an immediate consequence of the previous Theorem. □

Corollary 3.4. *We have the following:*

(a) *If $|X| < \mathfrak{p}$, then for any ideal $\mathcal{I}$ on ω and any sequence of real-valued functions defined on X, which $\mathcal{I}$QN-converges to some $f \in \mathbb{R}^X$, one can find its subsequence QN-converging to f.*

(b) *If $|X| \geq \mathfrak{c}$, then for any tall ideal $\mathcal{I}$ on ω there is a sequence of real-valued functions defined on X, which $\mathcal{I}$QN-converges to some $f \in \mathbb{R}^X$, but none of its subsequences QN-converges to f.*

Proof. First item in the case of non-tall ideals follows from Theorem 1.3. To prove the remaining parts it suffices to observe that $\mathfrak{p} \leq \mathtt{cov}^*(\mathcal{I}) \leq \mathfrak{c}$ for any tall ideal $\mathcal{I}$ on ω (cf. Remark 3.1). □

The anonymous referee of this paper had pointed out that item (b) of the above result can be strengthened in the following way.

Proposition 3.5. *If $|X| \geq \mathfrak{c}$, then there is a sequence (f_n) of real-valued functions defined on X such that for every tall ideal $\mathcal{I}$ on ω:*

- *the set $A(\mathcal{I}) = \{x \in X : f_n(x) \xrightarrow{\mathcal{I}} 0\}$ has cardinality $\mathfrak{c}$;*
- *$f_n \restriction_{A(\mathcal{I})} \xrightarrow{\mathcal{I}QN} 0$;*
- *there is $B(\mathcal{I}) \subseteq A(\mathcal{I})$ of cardinality $\mathtt{cov}^*(\mathcal{I})$ such that (f_n) has no subsequence QN-converging to 0 on $B(\mathcal{I})$.*

Proof. Fix a surjection $\phi \colon X \to [\omega]^\omega$ and define a sequence $(f_n) \subseteq \mathbb{R}^X$ by:

$$f_n(x) = \begin{cases} 1 & \text{if } n \in h(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then given any $x \in X$ we have $\{n \in \omega : |f_n(x)| \geq \varepsilon\} = h(x)$ for each $\varepsilon \in (0, 1)$. Therefore, $A(\mathcal{I}) = h^{-1}[\mathcal{I}]$ for every ideal $\mathcal{I}$ on ω and $f_n \restriction_{A(\mathcal{I})} \xrightarrow{\mathcal{I}\text{QN}} 0$. If $\mathcal{I}$ is tall, then it has cardinality $\mathfrak{c}$ (a tall ideal must have an infinite member and all infinite subsets of that member belong to the ideal as well). Hence, $|A(\mathcal{I})| = \mathfrak{c}$.

Let $\mathcal{A} \subseteq [\omega]^\omega$ be the family from the definition of $\mathtt{cov}^*(\mathcal{I})$. Find $B(\mathcal{I}) \subseteq A(\mathcal{I})$ of cardinality $\mathtt{cov}^*(\mathcal{I})$ such that $h[B(\mathcal{I})] = \mathcal{A}$. Now we will show that none of subsequences of (f_n) converges to 0 on $B(\mathcal{I})$. Fix any subsequence $(f_{n_m}) \subseteq (f_n)$. The set $S = \{n_m : m \in \omega\}$ is infinite, so one can find $x \in B(\mathcal{I})$ with $S \cap h(x)$ infinite (since $h[B(\mathcal{I})] = \mathcal{A}$). Then we have:

$$|\{m \in \omega : |f_{n_m}(x)| \geq 1\}| = |S \cap h(x)| = \omega.$$

□

4. Additivity of $\mathcal{I}$QN-spaces

Recall that for an ideal $\mathcal{I}$ on ω $\mathtt{add}(\mathcal{I}\text{QN-space})$ denotes the minimal cardinal κ such that there is a perfectly normal non-$\mathcal{I}$QN-space which can be expressed as a union of κ many $\mathcal{I}$QN-spaces.

Definition 4.1. *For an ideal $\mathcal{I}$ on ω denote by $\mathcal{P}_\mathcal{I}$ the family of all partitions of ω into sets belonging to $\mathcal{I}$. We define:*

$$\lambda(\mathcal{I}) = \min\left\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{P}_\mathcal{I} \ \wedge \ \forall_{(B_n)\in\mathcal{P}_\mathcal{I}} \ \exists_{(A_n)\in\mathcal{A}} \ \bigcup_{n\in\omega}\left(B_n \cap \bigcup_{k<n} A_k\right) \notin \mathcal{I}\right\}.$$

Remark 4.2. *Observe that $\lambda(\mathcal{I}) \leq \mathfrak{c}$ for every ideal $\mathcal{I}$ on ω, i.e., $\lambda(\mathcal{I})$ is well defined. Indeed, take $\mathcal{A} = \mathcal{P}_\mathcal{I}$. Then $|\mathcal{A}| \leq \mathfrak{c}$ and for each $(B_n) \in \mathcal{P}_\mathcal{I}$ the partition $(A_n) \in \mathcal{A}$ given by $A_0 = B_0 \cup B_1$ and $A_n = B_{n+1}$ for $n \in \omega \setminus \{0\}$ is such that:*

$$\bigcup_{n\in\omega}\left(B_n \cap \bigcup_{k<n} A_k\right) = \omega \setminus B_0 \notin \mathcal{I}.$$

Remark 4.3. *Observe that $\lambda(\mathcal{I}) \geq \omega_1$ for every ideal $\mathcal{I}$ on ω. Indeed, fix any $\{(A^m_n) \in \mathcal{P}_\mathcal{I} : \ m < \omega_1\}$ and define*

$$B_n = \left(\bigcup_{m\leq n}\bigcup_{k\leq n} A^m_k\right) \setminus \left(\bigcup_{m<n}\bigcup_{k<n} A^m_k\right)$$

for each $n \in \omega$. Then each B_n belongs to $\mathcal{I}$ (as a subset of a finite union of sets belonging to $\mathcal{I}$). Moreover, (B_n) is a partition of ω (as (A^0_n) is a partition of ω and $A^0_n \subseteq B_0 \cup \ldots \cup B_n$ for each $n \in \omega$). Now it suffices to observe that for each $m \in \omega$ we have:

$$\bigcup_{n\in\omega}\left(B_n \cap \bigcup_{k<n} A^m_k\right) = \bigcup_{n\leq m}\left(B_n \cap \bigcup_{k<n} A^m_k\right) \in \mathcal{I}.$$

Theorem 4.4. *The following are equivalent for any ideal $\mathcal{I}$ on ω:*

(a) *$\kappa < \lambda(\mathcal{I})$;*
(b) *if $X = \bigcup_{\alpha<\kappa} X_\alpha$ and $(f_n) \subseteq \mathbb{R}^X$ $\mathcal{I}$QN-converges to $f \in \mathbb{R}^X$ on each X_α, then (f_n) $\mathcal{I}$QN-converges to f on X.*

Proof. **(a)⇒(b):** Fix X and $X_\alpha \subseteq X$, for $\alpha < \kappa < \lambda(\mathcal{I})$, with $X = \bigcup_{\alpha<\kappa} X_\alpha$. Let $(f_n) \subseteq \mathbb{R}^X$ and $f \in \mathbb{R}^X$. Suppose that (f_n) $\mathcal{I}$QN-converges to f on each X_α with the witnessing sequence $(\varepsilon^\alpha_n) \subseteq (0,1)$. Define:

$$A^\alpha_k = \left\{n \in \omega : \ \frac{1}{k+2} \leq \varepsilon^\alpha_n < \frac{1}{k+1}\right\}$$

for each $n \in \omega$ and $\alpha < \kappa$. Then $(A^\alpha_n)_{n\in\omega}$, for any α, is a partition of ω and each A^α_k belongs to $\mathcal{I}$ since (ε^α_n) is $\mathcal{I}$-convergent to 0. As $\kappa < \lambda(\mathcal{I})$, there is $(B_n) \in \mathcal{P}_\mathcal{I}$ such that $\bigcup_{n\in\omega}\left(B_n \cap \bigcup_{k<n} A^\alpha_k\right) \in \mathcal{I}$ for each α.

Define a sequence $(\varepsilon_k) \subseteq (0,1)$ by:

$$\varepsilon_k = \frac{1}{n+1} \iff k \in B_n.$$

Then (ε_k) is $\mathcal{I}$-convergent to 0. We will show that it witnesses $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X.

Fix $x \in X$ and let $\alpha_0 < \kappa$ be such that $x \in X_{\alpha_0}$. We have:

$$\{k \in \omega :\ |f_k(x) - f(x)| \geq \varepsilon_k\} \subseteq$$
$$\subseteq \{k \in \omega :\ \varepsilon_k < \varepsilon_k^{\alpha_0}\} \cup \{k \in \omega :\ |f_k(x) - f(x)| \geq \varepsilon_k^{\alpha_0}\}.$$

The latter set belongs to $\mathcal{I}$ since $(\varepsilon_n^{\alpha_0})$ witnesses $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X_{α_0}. Now it suffices to show that $\{k \in \omega :\ \varepsilon_k < \varepsilon_k^{\alpha_0}\} \in \mathcal{I}$. Indeed, we have:

$$\{k \in \omega :\ \varepsilon_k < \varepsilon_k^{\alpha_0}\} = \bigcup_{n\in\omega} \left(B_n \cap \bigcup_{k<n} A_k^{\alpha_0} \right) \in \mathcal{I}.$$

(b)⇒(a): Fix any set X of cardinality at least $\lambda(\mathcal{I})$, let $\phi\colon \lambda(\mathcal{I}) \to X$ be an injection and define $X_\alpha = \{\phi(\alpha)\}$ for each $\alpha < \lambda(\mathcal{I})$. Set also a family $\{(A_n^\alpha)_{n\in\omega} :\ \alpha < \lambda(\mathcal{I})\} \subseteq \mathcal{P}_\mathcal{I}$ such that for every $(B_n) \in \mathcal{P}_\mathcal{I}$ one can find $\alpha < \lambda(\mathcal{I})$ with $\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{k<n} A_k^\alpha\right) \notin \mathcal{I}$. Define a sequence of functions $(f_n) \in \mathbb{R}^X$ by:

$$f_k(x) = \begin{cases} \frac{1}{n+1} & \text{if } k \in A_n^\alpha \text{ and } x = \phi(\alpha) \text{ for some } \alpha < \lambda(\mathcal{I}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \xrightarrow{\mathcal{I}\text{QN}} 0$ on each X_α as $(A_n^\alpha)_{n\in\omega} \in \mathcal{P}_\mathcal{I}$.

We will show that (f_n) does not $\mathcal{I}$QN-converge to 0 on X. Suppose to the contrary that $f_n \xrightarrow{\mathcal{I}\text{QN}} 0$ on X and it is witnessed by some sequence $(\varepsilon_k) \subseteq (0,1)$. Define:

$$B_n = \left\{ k \in \omega :\ \frac{1}{n+2} \leq \varepsilon_k < \frac{1}{n+1} \right\}$$

for all $n \in \omega$. Then $(B_n) \in \mathcal{P}_\mathcal{I}$. Hence, by the definition of $\{(A_n^\alpha)_{n\in\omega} :\ \alpha < \lambda(\mathcal{I})\}$, there is $\alpha_0 < \lambda(\mathcal{I})$ such that $C = \bigcup_{n\in\omega} \left(B_n \cap \bigcup_{k<n} A_k^{\alpha_0}\right) \notin \mathcal{I}$. We will show that $C \subseteq \{k \in \omega :\ |f_k(\phi(\alpha_0))| \geq \varepsilon_k\}$, which will end the proof. Fix any $m \in C$. Then there is $n \in \omega$ such that $m \in B_n \cap \bigcup_{k<n} A_k^{\alpha_0}$. Therefore, $\varepsilon_m < \frac{1}{n+1}$ and $f_m(\phi(\alpha_0)) \geq \frac{1}{n+1}$. Hence, $m \in \{k \in \omega :\ |f_k(\phi(\alpha_0))| \geq \varepsilon_k\}$. □

From the above Proposition we can easily derive a connection between $\lambda(\mathcal{I})$ and $\mathtt{add}(\mathcal{I}\text{QN-space})$.

Corollary 4.5. *$\lambda(\mathcal{I}) \leq \mathtt{add}(\mathcal{I}\text{QN-space})$ for every ideal $\mathcal{I}$ on ω.*

Proof. Obvious. □

Remark 4.6. *Notice that $\lambda(\mathcal{I})$ and $\mathtt{add}(\mathcal{I}\text{QN-space})$ are not the same. Indeed, by Theorem 1.4, any topological space is a $(Fin \otimes Fin)$QN-space. Therefore, it does not make sense to consider additivity of $(Fin \otimes Fin)$QN-spaces. However, $\lambda(Fin \otimes Fin) \leq \mathfrak{c}$ by Remark 4.2.*

Now we proceed to obtaining a lower and upper bounds for $\mathtt{add}(\mathcal{I}\text{QN-space})$ and $\lambda(\mathcal{I})$.

Corollary 4.7. *We have:*

(a) *$\omega_1 \leq \lambda(\mathcal{I}) \leq \mathtt{add}(\mathcal{I}\text{QN-space}) \leq \mathfrak{d}$ for every weak P-ideal $\mathcal{I}$ on ω;*
(b) *$\omega_1 \leq \mathtt{add}(\mathcal{I}\text{QN-space}) \leq \mathfrak{b}$ for every ideal $\mathcal{I}$ on ω which is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal;*
(c) *$\lambda(\mathcal{I}) = \mathtt{add}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$ for every P-ideal $\mathcal{I}$ on ω which is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal. In particular, $\lambda(\mathcal{I}) = \mathtt{add}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$ for every analytic P-ideal generated by an unbounded submeasure.*

Proof. **(a):** In fact, for every weak P-ideal $\mathcal{I}$ on ω we have the following sequence of inequalities:

$$\omega_1 \leq \lambda(\mathcal{I}) \leq \mathtt{add}(\mathcal{I}\text{QN-space}) \leq \mathtt{non}(\mathcal{I}\text{QN-space}) \leq \mathfrak{d}.$$

Indeed, the first inequality is Remark 4.3, the second one is Corollary 4.5, the third inequality is obvious (as all singleton spaces are $\mathcal{I}$QN-spaces for every ideal $\mathcal{I}$) and the last one follows from item (a) of Theorem 2.7.

(b): It suffices to use item (b) of Theorem 2.7 instead of (a) in the above considerations.

(c): The first part is a combination of item (b) and [11, Theorem 2.2] stating that $\lambda(\mathcal{I}) \geq \mathfrak{b}$ for all P-ideals $\mathcal{I}$. Analytic P-ideals generated by unbounded submeasures are $\leq_K$-below some $\mathtt{F}_\sigma$ ideal (cf. the proof of Corollary 2.8) and $\mathtt{F}_\sigma$ P-ideals are generated by unbounded submeasures. □

By Theorem 1.3 we have $\mathtt{add}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$ for all non-tall ideals $\mathcal{I}$. By the last item of Corollary 4.7 we also have $\mathtt{add}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$ for analytic P-ideals $\mathcal{I}$ on ω which are $\leq_K$-below some $\mathtt{F}_\sigma$ ideal. We want to end this section with an example of a class of tall ideals $\mathcal{I}$ which are not P-ideals and satisfy $\mathtt{add}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$.

Proposition 4.8. *For any ideal $\mathcal{I}$ on ω we have $\lambda(\mathcal{I} \otimes \emptyset) = \lambda(\mathcal{I})$.*

Proof. Define $f\colon \mathcal{P}_\mathcal{I} \to \mathcal{P}_{\mathcal{I}\otimes\emptyset}$ by $f((A_n)) = (A_n^f) = (A_n \times \omega)$ and $g\colon \mathcal{P}_{\mathcal{I}\otimes\emptyset} \to \mathcal{P}_\mathcal{I}$ by $g((A_n)) = (A_n^g)$, where

$$i \in A_n^g \iff n = \min\{k \in \omega :\ A_k \cap (\{i\} \times \omega) \neq \emptyset\}.$$

First we will show that $\lambda(\mathcal{I} \otimes \emptyset) \leq \lambda(\mathcal{I})$. Let $\mathcal{A} \subseteq \mathcal{P}_\mathcal{I}$ be as in the definition of $\lambda(\mathcal{I})$. We claim that $f[\mathcal{A}] \subseteq \mathcal{P}_{\mathcal{I}\otimes\emptyset}$ is as needed. Fix any $(B_n) \in \mathcal{P}_{\mathcal{I}\otimes\emptyset}$. Then there is $(A_n) \in \mathcal{A}$ with $\bigcup_{n\in\omega} \left(B_n^g \cap \bigcup_{k<n} A_k\right) \notin \mathcal{I}$. Note that for each $i \in B_n^g$ there is $j \in \omega$ with $(i,j) \in B_n$. If additionally $i \in \bigcup_{k<n} A_k$, then $(i,j) \in \bigcup_{k<n} A_k^f$. Hence, $\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{k<n} A_k^f\right) \notin \mathcal{I} \otimes \emptyset$.

Now we will show that $\lambda(\mathcal{I}) \leq \lambda(\mathcal{I} \otimes \emptyset)$. Let $\mathcal{A} \subseteq \mathcal{P}_{\mathcal{I}\otimes\emptyset}$ be as in the definition of $\lambda(\mathcal{I} \otimes \emptyset)$. We claim that $g[\mathcal{A}] \subseteq \mathcal{P}_\mathcal{I}$ is as needed. Fix any $(B_n) \in \mathcal{P}_\mathcal{I}$. Then there is $(A_n) \in \mathcal{A}$ with $\bigcup_{n\in\omega} \left(B_n^f \cap \bigcup_{k<n} A_k\right) \notin \mathcal{I} \otimes \emptyset$. Hence,

$$\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{k<n} A_k^g\right) = \bigcup_{n\in\omega} \left\{i \in B_n :\ \exists_{j\in\omega}\ (i,j) \in \bigcup_{k<n} A_k\right\} \notin \mathcal{I}.$$

□

Lemma 4.9. *$\mathcal{I} \otimes \emptyset \leq_{KB} \mathcal{I}$ for any ideal $\mathcal{I}$ on ω.*

Proof. We claim that the function $f\colon \omega \to \omega \times \omega$ given by $f(n) = (n,0)$, for $n \in \omega$, witnesses $\mathcal{I} \otimes \emptyset \leq_{KB} \mathcal{I}$. Indeed, f is finite-to-one (even one-to-one) and given any $A \in \mathcal{I} \otimes \emptyset$ we have:

$$f^{-1}[A] \subseteq \{n \in \omega :\ A \cap (\{n\} \times \omega) \neq \emptyset\} \in \mathcal{I}.$$

□

Corollary 4.10. *Let $\mathcal{I}$ be a P-ideal and $\mathcal{J}$ be any ideal on ω isomorphic to $\mathcal{I} \otimes \emptyset$. Then we have:*

(a) *if* $\mathtt{non}(\mathcal{I}\textit{QN-space}) = \mathfrak{b}$, *then*

$$\mathtt{add}(\mathcal{J}\textit{QN-space}) = \mathtt{non}(\mathcal{J}\textit{QN-space}) = \mathfrak{b};$$

(b) *if $\mathcal{I}$ is $\leq_K$-below some $\mathtt{F}_\sigma$ ideal (in particular, if $\mathcal{I}$ is generated by an unbounded submeasure), then*

$$\texttt{add}(\mathcal{J}\,\textit{QN-space}) = \texttt{non}(\mathcal{J}\,\textit{QN-space}) = \texttt{non}(\mathcal{J}\,\textit{wQN-space}) = \mathfrak{b}.$$

Proof. By [11, Theorem 2.2], $\lambda(\mathcal{I}) \geq \mathfrak{b}$ for each P-ideal $\mathcal{I}$. Hence, by Proposition 4.8, we have:

$$\mathfrak{b} \leq \lambda(\mathcal{I}) = \lambda(\mathcal{J}) \leq \texttt{add}(\mathcal{J}\text{QN-space}) \leq \texttt{non}(\mathcal{J}\text{QN-space}) \leq \texttt{non}(\mathcal{J}\text{wQN-space}).$$

Theorem 1.2 and Lemma 4.9 give us $\texttt{non}(\mathcal{J}\text{QN-space}) \leq \texttt{non}(\mathcal{I}\text{QN-space}) = \mathfrak{b}$. This proves part (a). To show part (b) observe that $\mathcal{J}$ is $\leq_K$-below the same $\mathtt{F}_\sigma$ ideal as $\mathcal{I}$ (hence, $\texttt{non}(\mathcal{J}\text{wQN-space}) = \mathfrak{b}$ by Theorem 2.7). Indeed, this follows from transitivity of the Katětov order, as $\mathcal{J} \leq_K \mathcal{I} \otimes \emptyset \leq_K \mathcal{I}$ by Lemma 4.9, and the fact that $\mathcal{J}$ and $\mathcal{I} \otimes \emptyset$ are isomorphic. □

Remark 4.11. *Note that the equality $\texttt{add}(\mathcal{J}\,\textit{QN-space}) = \mathfrak{b}$ in Corollary 4.10 cannot be derived from Corollary 4.7, as the ideal $\mathcal{J}$ from Corollary 4.10 is never a P-ideal. Indeed, it suffices to show that $\mathcal{I} \otimes \emptyset$ is not a P-ideal, for any ideal $\mathcal{I}$ on ω, and this is witnessed by the sequence $(\{n\} \times \omega) \subseteq \mathcal{I} \otimes \emptyset$. What is more, it is easy to show that $\mathcal{I} \otimes \emptyset$ is tall if and only if $\mathcal{I}$ is tall. Hence, if $\mathcal{I}$ is a tall analytic P-ideal generated by an unbounded submeasure (this is the case for instance for each summable ideal or a simple density ideal which is not an Erdős-Ulam ideal – see Subsection 1.2), then item (b) of Corollary 4.10 gives us $\texttt{add}(\mathcal{J}\,\textit{QN-space}) = \mathfrak{b}$ for a tall ideal $\mathcal{J}$ which is not a P-ideal.*

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