TOPOLOGICAL REPRESENTATIONS

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ABSTRACT. This paper studies the combinatorics of ideals which recently appeared in ergodicity results for analytic equivalence relations. The ideals have the following topological representation. There is a separable metrizable space X, a σ-ideal I on X and a dense countable subset D of X such that the ideal consists of those subsets of D whose closure belongs to I. It turns out that this definition is indepedent of the choice of D. We show that an ideal is of this form if and only if it is dense and countably separated. The latter is a variation of a notion introduced by Todorčević for gaps. As a corollary, we get that this class is invariant under the Rudin–Blass equivalence. This also implies that the space X can be always chosen to be compact so that I is a σ-ideal of compact sets. We compute the possible descriptive complexities of such ideals and conclude that all analytic equivalence relations induced by such ideals are $\mathbf{\Pi}^0_3$. We also prove that a coanalytic ideal is an intersection of ideals of this form if and only if it is weakly selective.

1. Introduction

The aim of this paper is to reveal a connection between the structure of ideals on countable sets and ideals of compact sets in Polish spaces. A family of subsets of a given set is an *ideal* if it is closed under taking subsets and finite unions. We always assume that an ideal of subsets of a set S contains all singletons $\{s\}$ for $s \in S$, (i.e. an ideal J is an ideal of subsets of $\bigcup J$) and does not contain S (i.e. it is a proper ideal). Given an ideal J, we say that a set is *J-positive* if it does not belong to J. Sometimes, we write J^+ for the family of J-positive sets and co-J for the dual filter. Throughout this paper we often identify subsets of ω with elements of 2^ω via the characteristic functions. Thus, for example, given an ideal of subsets of ω we define its descriptive complexity as if it were a subset of 2^ω. On the other hand, given an ideal of compact subsets of a given Polish space X we refer to its descriptive complexity in the Vietoris space $K(X)$.

The study of definable ideals of compact sets has become a classical subject in descriptive set theory. A well-known result of Kechris, Louveau and

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Woodin [17] and Dougherty, Kechris and Louveau (see [15]) says that an analytic ideal of compact sets is a σ-ideal if and only if it is $\mathbf{\Pi}^0_2$. The descriptive complexity of more complicated ideals of compact sets is the subject of a trichotomy theorem proved by Matheron, Solecki and Zelený [20]. Recently, Solecki [26] described a special class of $\mathbf{\Pi}^0_2$ σ-ideals of compact sets and proved several structure theorems representing ideals in that class via the meager ideal (see also [18]).

The structure of ideals on countable sets has been studied from a different perspective but some results reveal the similarities. An analogy to the Kechris–Louveau–Woodin theorem appears in the work of Solecki [25] on analytic P-ideals, where it is shown that if a P-ideal is analytic, then its descriptive complexity is $\mathbf{\Pi}^0_3$. Solecki also shows [24, Corollary 3.4] that if J is an analytic P-ideal then it is either $\mathbf{\Sigma}^0_2$ or $\mathbf{\Pi}^0_3$-complete. The structure of ideals on ω is often described in terms of the *Rudin–Blass order*. Given two ideals J, K on ω we say that J is *Rudin–Blass below* K and write $J \leq_{\mathrm{RB}} K$ if there is a finite-to-one function $f : \omega \to \omega$ such that $a \in J$ if and only if $f^{-1}(a) \in K$, for every $a \subseteq \omega$. The Jalali-Naini–Mathias–Talangrand theorem [2, Theorem 4.1.2] then says that every ideal with the Baire property is Rudin–Blass above the ideal Fin of finite sets.

A connection between ideals of compact sets and ideals on countable sets that appears in this paper uses the following operation. Suppose X is a separable metrizable space and I is a σ-ideal on X that contains all singletons. Given a dense countable set $D \subseteq X$, define the ideal J_I on D as the family $\{a \subseteq D : \mathrm{cl}(a) \in I\}$. Obviously, J_I depends only on the family of closed sets that belong to I. In principle, J_I also depends on the set D, which is equal to $\bigcup J_I$, but we will see (Proposition 2.1) that, up to isomorphism, this definition is independent of the choice of D. The ideals of the form J_I have been recently studied in [23] and used in canonization (see [14]) of smooth equivalence relations for σ-ideals generated by closed sets. Given an ideal J on a countable set E we say that J *has a topological representation* if there are I, D, X as above and a bijection $\rho : E \to D$ such that $J = \{a \subseteq E : \rho(a) \in J_I\}$. In such a case we say that J is *represented by* I. We also say then that J is *represented on* X.

Two examples of ideals with topological representations have been studied by Farah and Solecki [9], who showed that there are at least two isomorphism types of such ideals (namely that the ideals represented by the meager sets and by the meager null sets are not isomorphic).

The study of ideals on ω is closely connected and largely motivated by the study of equivalence relations on 2^ω given in the following way. Given an ideal J on ω we write E_J for the equivalence relation on 2^ω with $x \; E_J \; y$ if $x \Delta y \in J$. Rosendal [22] proved that any Borel equivalence relation is Borel-reducible to one of the form E_J. A motivation for the results in this paper is a recent work of Zapletal [28], who shows that if J has a topological representation, then the equivalence relation E_J has the following ergodicity property. First, every Borel homomorphism from a turbulent equivalence relation F to E_J maps a comeager set to a single E_J-equivalence class. Second, if J is represented by a $\mathbf{\Pi}^0_2$ σ-ideal, then every homomorphism from E_J to an equivalence relation classifiable by countable structures maps a measure 1 set to a single equivalence class. This is in contrast with the turbulence dichotomy of Hjorth [11]. Note that if J has a topological representation, then it is not a P-ideal and hence E_J is not induced by a Polish group action by the Solecki characterization of Polishable P-ideals [25, Corollary 4.1]. We would like to note here that recently a different notion of representations of ideals has been considered in [4] and the ideals considered there are actually P-ideals.

The main result of this paper establishes a combinatorial characterization of ideals which admit topological representations. One of the necessary conditions says that the ideal is *dense*, i.e. any infinite set contains an infinite subset that belongs to the ideal. The other condition is a variation of Todorčević's notion of countably separated gaps [27] (see also [6, 8, 1]). We say that an ideal J on a countable set D is *countably separated* if there is a countable family $\{a_n : n < \omega\}$ of subsets of D such that for any $a, b \subseteq D$ with $a \in J$ and $b \notin J$ there is $n \in \omega$ with $a \cap a_n = \emptyset$ and $b \cap a_n \notin J$. In such a case we say that the family $\{a_n : n < \omega\}$ *separates* J. We prove the following characterization.

Theorem 1.1. *For any ideal J on a countable infinite set the following are equivalent:*

(i) *J is dense and countably separated,*
(ii) *J has a topological representation,*
(iii) *J has a topological representation on the Cantor space.*

As a corollary we get the following

Corollary 1.2. *The class of ideals which have topological representations is invariant under the Rudin–Blass equivalence.*

Proof. It is enough to show that if $J \leq_{\mathrm{RB}} K$ and K is countably separated, then J is countably separated, and if J is dense, then K is dense. Let $f : \omega \to \omega$ be a Rudin–Blass reduction witnessing $J \leq_{\mathrm{RB}} K$.

Suppose first that K is countably separated by $\{c_n : n < \omega\}$ and let $d_n = f''c_n$. We claim that d_n's witness that J is countably separated. Indeed, take $a \in J$ and $b \notin J$. Then $a' = f^{-1}(a) \in K$ and $b' = f^{-1}(b) \notin K$. Pick n such that $c_n \cap a' = \emptyset$ and $c_n \cap b' \notin K$. Then $d_n \cap a = \emptyset$ and $d_n \cap b \notin K$.

Suppose now that J is dense and let $b \subseteq \omega$ be infinite. Since f is finite-to-one, $b' = f''b$ is also infinite and hence there is $c' \subseteq b'$ with $c' \in J$. Let $c = f^{-1}(c') \cap b$ and note that c is an infinite subset of b which belongs to K, as a subset of $f^{-1}(c')$. $\square$

Corollary 1.2 in particular implies that ideals with topological representations are invariant under $\equiv^{++}_{\mathrm{RB}}$ (see [13, Section 3.2]) and hence the class of equivalence relations of the form E_J, for J with a topological representation, is invariant under additive Borel reductions, by a result of Farah [7].

Theorem 1.1 says that the space on which an ideal can be represented can always be chosen to be the Cantor space. It turns out that there is also some control over the closed sets generating the σ-ideal.

Corollary 1.3. *If J has a topological representation, then it is represented on the Cantor space by a σ-ideal generated by a family of compact nowhere dense sets.*

In particular, the family representing an ideal can be chosen to consist of compact sets. For definable ideals, this implies computations of possible descriptive complexities.

Theorem 1.4. *If an ideal J has a topological representation and it is analytic, then it is $\mathbf{\Pi}^0_3$-complete. In particular, if E_J is analytic, then it is $\mathbf{\Pi}^0_3$.*

This gives an analogoue of the Kechris–Louveau–Woodin dichotomy.

Corollary 1.5. *If a coanalytic ideal J has a topological representation, then J is either $\mathbf{\Pi}^0_3$-complete, or $\mathbf{\Pi}^1_1$-complete. J is $\mathbf{\Pi}^0_3$-complete if and only if it is represented by a $\mathbf{\Pi}^0_2$ ideal of compact sets.*

Besides the descriptive complexity, there is one more combinatorial condition that determines the structure of ideals which have topological representations. An ideal J is *weakly selective* if for every $b \notin J$ and a function $f : b \to \omega$ there is a J-positive subset a of b such that $f \upharpoonright a$ is either one-to-one or constant. Equivalently, J is weakly selective if any partition of a J-positive set into sets in J admits a J-positive selector. Weakly selective ideals have been studied by several authors (see Farah [5] or Baumgartner and Laver [3]) and [23, Proposition 4.3] shows that if J has a topological representation, then it is weakly selective. Here we prove the following characterization.

Theorem 1.6. *Let J be a coanalytic ideal. The following are equivalent:*

(i) *J is weakly selective,*
(ii) *J is an intersection of a family of ideals with topological representations.*

The paper is organized as follows. A preliminary discussion and a proof of Theorem 1.1 are given in Section 2. Corollary 1.3 is proved in Section 3. Theorem 1.4 together with Corollary 1.5 are proved in Section 4. Section 5 contains a proof of Theorem 1.6.

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2. A characterization of ideals with topological representations

It is fairly easy to see that if J has a topological representation, then it is also represented on a compact metric space. Indeed, if J is represented on X via a σ-ideal I of closed sets, then let $\hat{X}$ be a metric compactification of X and let $\hat{I}$ be the σ-ideal of compact sets on $\hat{X}$ generated by the sets $\mathrm{cl}(K)$ (the closure is taken in $\hat{X}$) for closed $K \subseteq X$ with $K \in I$ as well as the singletons $\{x\}$ for $x \in \hat{X} \setminus X$. Then J_I is represented on $\hat{X}$ via $\hat{I}$ as witnessed by the same dense set $D \subseteq \hat{X}$. In this section we show that X can be always chosen to be the Cantor set.

Before we do that, let us comment on the choice of the dense set D in the definition of the ideals J_I.

Proposition 2.1. *Given a σ-ideal I on a separable metric space X and two dense countable sets D and E in X, if $J = \{a \subseteq D : \mathrm{cl}(a) \in I\}$ and $K = \{a \subseteq E : \mathrm{cl}(a) \in I\}$, then J and K are isomorphic.*

Proof. First, note that by the remarks at the beginning of this section, we can assume that X is compact. Second, assume without loss of generality that X either has no isolated points or has infinitely many isolated points (if there are finitely many isolated points, then remove them from both D and E). Throughout this proof cl denotes the closure in X.

Case 1. First assume that X has no isolated points. Using a back-and-forth argument, enumerate $D = \{d_n : n < \omega\}$ and $E = \{e_n : n < \omega\}$ so that the distance of d_n and e_n is smaller than $1/n$. Now for $a \subseteq \omega$ write $\mathrm{cl}_D(a) = \mathrm{cl}(\{d_n : n \in a\})$ and $\mathrm{cl}_E(a) = \mathrm{cl}(\{e_n : n \in a\})$. To see that J and K are isomorphic, it is enough to show that $\mathrm{cl}_D(a)$ belongs to I if and only if $\mathrm{cl}_E(a)$ belongs to I. Note that

(1) $\mathrm{cl}_D(a) \subseteq \mathrm{cl}_E(a) \cup \{d_n : n \in a\}$,
(2) $\mathrm{cl}_E(a) \subseteq \mathrm{cl}_D(a) \cup \{e_n : n \in a\}$.

Here, (1) follows from the fact that if x belongs to $\mathrm{cl}_D(a)$ and is not one of the d_n's (for $n \in a$), then there is an infinite subsequence of d_n's (indexed with elements of a) that converges to x. Since e_n is $(1/n)$-close to d_n, there is also an infinite subsequence of e_n's (with the same index set) converging to x. (2) follows by symmetry. Now, (1) and (2) imply that $\mathrm{cl}_D(a)$ and $\mathrm{cl}_E(a)$ can differ by an at most countable set. Since the singletons belong to I (we always assume that ideals contain all singletons), it follows that $\mathrm{cl}_D(a) \in I$ if and only if $\mathrm{cl}_E(a) \in I$.

Case 2. Now assume that X has infinitely many isolated points. Write $E^0 = D^0$ for the set of isolated points of X (note that $E^0 = D^0 \subseteq E, D$).

Now proceed similarly as in Case 1. Write d_X for a metric on X. Using a back-and-forth argument (at even stages taking care of D and at odd stages taking care of E), enumerate $D = \{d_n : n < \omega\}$ and $E = \{e_n : n < \omega\}$ so that

- if n is even and d_n is non-isolated, then
$$d_X(d_n, e_n) < 1/n,$$
- if n is even and d_n is isolated, then
$$d_X(d_n, e_n) < 2\inf\{d_X(d_n, e) : e \in E^0 \setminus \{d_n\} \setminus \{e_i : i < n\}\}$$
and $e_n \in E^0$,
- if n is odd and e_n is non-isolated, then
$$d_X(d_n, e_n) < 1/n,$$
,
- if n is odd and e_n is isolated, then
$$d_X(d_n, e_n) < 2\inf\{d_X(d, e_n) : d \in D^0 \setminus \{e_n\} \setminus \{d_i : i < n\}\},$$
and $d_n \in D^0$.

Again, for $a \subseteq \omega$ write $\mathrm{cl}_D(a) = \mathrm{cl}(\{d_n : n \in a\})$ and $\mathrm{cl}_E(a) = \mathrm{cl}(\{e_n : n \in a\})$. To see that J and K are isomorphic, it is again enough to show that $\mathrm{cl}_D(a)$ belongs to I if and only if $\mathrm{cl}_E(a)$ belongs to I. We claim that

(1) $\mathrm{cl}_D(a) \subseteq \mathrm{cl}_E(a) \cup \{d_n : n \in a\} \cup D^0$,
(2) $\mathrm{cl}_E(a) \subseteq \mathrm{cl}_D(a) \cup \{e_n : n \in a\} \cup E^0$.

To see (1) assume that x belongs to $\mathrm{cl}_D(a)$ and is neither one of the d_n's (for $n \in a$) nor an isolated point. Then there is an infinite subsequence of $\{d_n : n \in \omega\}$ (indexed with elements of a) that converges to x. Say that the sequence is indexed with k_n's (with each k_n in a). There are two subcases.

Subcase 1. Assume infinitely many k_n's are even. But then $d_{k_n} \to x$ implies that $e_{k_n} \to x$ by the first two items above.

Subcase 2. Assume infinitely many k_n's are odd. We still claim that $e_{k_n} \to x$. Otherwise, by compactnesss there is a subsequence of e_{k_n} converging to some $y \neq x$. But then by the last two items above, we get that the corresponding subsequence of d_{k_n} converges to y as well, contradiction.

Now, in both subcases, we get that x belongs to $\mathrm{cl}_E(a)$, as needed. The claim (2) follows by symmetry. Finally, (1) and (2) again imply that $\mathrm{cl}_D(a)$ and $\mathrm{cl}_E(a)$ can differ by an at most countable set and we are done as before. □

Now we will prove Theorem 1.1. Let us first comment on the sharpness of condition (i) in that theorem: neither being dense nor countably separated alone implies that the ideal has a topological representation.

To see that, first consider the ideal $\emptyset \times \mathrm{Fin} = \{a \subseteq \omega \times \omega : \forall n \in \omega\ a_n \in \mathrm{Fin}\}$, where $a_n = \{m \in \omega : (n, m) \in a\}$. This ideal is countably separated, by the sets $c_{n,k} = \{(n, m) \in \omega \times \omega : m > k\}$ but it is clearly not dense.

On the other hand, consider the ideal $\mathrm{Fin} \times \mathrm{Fin} = \{a \subseteq \omega \times \omega : \{n \in \omega : a_n \notin \mathrm{Fin}\} \in \mathrm{Fin}\}$. The ideal $\mathrm{Fin} \times \mathrm{Fin}$ is not weakly selective, as witnessed by the projection function $(n, m) \mapsto n$. By [23, Proposition 4.3], an ideal which has a topological representation is weakly selective. Thus, $\mathrm{Fin} \times \mathrm{Fin}$

does not have a topological representation. On the other hand, it is clearly dense. Thefore, by Theorem 1.1 it cannot be countably separated.

Below, we fix a metric d on 2^ω and for $\varepsilon > 0$ and $A \subseteq 2^\omega$, write

$$\mathrm{ball}(\varepsilon, A) = \{x \in 2^\omega : \exists y \in A \quad d(x,y) < \varepsilon\}.$$

Proof of Theorem 1.1. The implication (iii)⇒(ii) is obvious.

(ii)⇒(i) Suppose J is represented on X and let $D \subseteq X$ be countable infinite and dense, I be a σ-ideal such that $J = J_I$. First note that J_I is dense. Indeed, take an infinite $a \notin J_I$. Then $\mathrm{cl}(a) \notin I$, so in particular $\mathrm{cl}(a)$ is uncountable. Let $x \in \mathrm{cl}(a) \setminus a$ and pick a sequence $\langle x_n : n \in \omega\rangle$ of elements of a converging to x. Then $b = \{x_n : n < \omega\}$ is an infinite subset of a, which is in J_I since $\mathrm{cl}(b) = b \cup \{x\}$ is countable and hence in I.

To see that J_I is countably separated, fix a countable basis $\{U_n : n < \omega\}$ of X and let $c_n = U_n \cap D$. We claim that $\{c_n : n < \omega\}$ witnesses that J is countably separated. Indeed, let $a, b \subseteq D$ be such that $a \in J_I, b \notin J_I$. Then $\mathrm{cl}(a) \in I$ and $\mathrm{cl}(b) \notin I$. By countable additivity of I, there exists n such that $U_n \cap \mathrm{cl}(a) = \emptyset$ and $U_n \cap \mathrm{cl}(b) \notin I$. Then clearly $c_n \cap a = \emptyset$ and $c_n \cap b$ is J-positive since $\mathrm{cl}(c_n \cap b)$ contains $U_n \cap \mathrm{cl}(b)$.

(i)⇒(iii) Suppose now that J is countably separated and dense. Assume that J is an ideal on ω. We will first show that the family $\{c_n : n \in \omega\}$ witnessing that J is countably separated can be improved a little. We say that a family of subsets of ω *separates points* if for each $n \neq m \in \omega$ there is a set a in that family such that $n \in a$ and $m \notin a$.

Lemma 2.2. *If J is countably separated, then there is a family witnessing that J is countably separated, which separates points and is such that all Boolean combinations of its elements are either infinite or empty.*

Proof. Let $\{c_n : n \in \omega\}$ be a family witnessing that J is countably separated. Enumerate all pairs of distinct natural numbers as $\langle (k_n, l_n) : n \in \omega\rangle$. We will construct a family $\{d_n : n \in \omega\}$ of subsets of ω such that for each n the following is true

(a) if $n = 2m$ is even, then d_n is a subset of c_m such that $c_m \setminus d_n \in J$;
(b) if $n = 2m+1$ is odd, then $k_m \in d_n$ and $l_n \notin d_n$;
(c) all Boolean combinations of d_i for $i \leq n$ are infinite or empty.

Notice that such a family will also witness that J is countably separated by (a). It will separate points by (b) and have all Boolean combinations either empty of infinite by (c). Hence, $\{d_n : n < \omega\}$ will be the required family.

To construct the sets d_n inductively, we start with $d_0 = c_0$. Suppose that d_k for $k < n$ have been constructed. All Boolean combination of $\{d_k : k < n\}$ define a finite partition $\{a_k : k < k_n\}$ of ω into infinite subsets.

Case 1. Suppose that $n = 2m$ is even. For each $k < k_n$ we define a set $e_k \subset a_k \cap c_m$ in the following way. There are three possibilities:

- if $a_k \cap c_m \in J$, then $e_k = \emptyset$;
- if $a_k \cap c_m \notin J$ and $a_k \setminus c_m$ is infinite, then $e_k = a_k \cap c_m$;

- if $a_k \cap c_m \notin J$ and $a_k \setminus c_m$ is finite, then find an infinite subset $e'_k \in J$ of $a_k \cap c_m$ (using the fact that J is dense) and define $e_k = (a_k \cap c_m) \setminus e'_k$.

The set $d_n = \bigcup_{k<k_n} e_k$ is a subset of c_m such that $c_m \setminus d_n \in J$. Also, d_n is either empty or both infinite and coinfinite in every a_k, therefore it is as needed.

Case 2. Suppose that $n = 2m+1$ is odd. There is $k < k_n$ such that $k_m \in a_k$. Let d_n be any infinite subset of a_k such that $k_m \in d_n$, $l_m \notin d_n$ and $a_k \setminus d_n$ is infinite. Then d_n separates the pair (k_m, l_m) and in each a_k it is either empty or infinite and coinfinite, therefore it is as needed. □

We can now assume that a family $\{c_n : n < \omega\}$ witnessing that J is countably separated is as in Lemma 2.2. Define a topology τ on ω by letting all c_n's be clopen basic sets. Consider the map

$$\omega \ni k \mapsto x_k = \{n \in \omega : k \in c_n\} \in 2^\omega$$

and let $C = \mathrm{cl}(\{x_k : k \in \omega\}) \subseteq 2^\omega$. Since all Boolean combinations of c_n's are either empty or infinite, C has no isolated points. Thus, C is a Cantor set and $D = \{x_k : k \in \omega\}$ is dense in C. Via this embedding, we treat now J as an ideal on D.

Define an ideal of $K(2^\omega)$ by $I = \{A \in K(2^\omega) : \exists a \in J \quad A \subset \mathrm{cl}(a)\}$. It turns out that I is a σ-ideal on $K(2^\omega)$.

Lemma 2.3. *I is a σ-ideal of compact sets.*

Proof. Suppose it is not a σ-ideal. Fix a metric d on 2^ω of diameter ≤ 1. The metric notions below refer to the metric d. If I is not a σ-ideal, then, by [20, Lemma 2.1] (this observation is also implicit in [15]), there is a sequence of sets $B_n \in I$ and $A \in I$ such that B_n converge to A in the Hausdorff metric and $A \cup \bigcup_n B_n \notin I$. Find $a \subseteq D$ such that $A \subseteq \mathrm{cl}(a)$ and $a \in J$ and for each n find $b_n \subseteq D$ such that $b_n \in J$, $B_n \subseteq \mathrm{cl}(b_n)$ and $\mathrm{cl}(b_n)$ is contained in $\mathrm{ball}(1/n, B_n)$. Write $b = a \cup \bigcup_n b_n$ and note that $b \notin J$ because $A \cup \bigcup_n B_n \subseteq \mathrm{cl}(b)$. Since J is countably separated by c_n's, there is n such that $a \cap c_n = \emptyset$ and $b \cap c_n \notin J$. Now, since c_n's are clopen on D, we get a clopen set $C \subseteq 2^\omega$ such that $a \subseteq C$ and $C \cap c_n = \emptyset$. Let $\varepsilon > 0$ be such that $\mathrm{ball}(\varepsilon, A) \subseteq C$. By the definition of b_k's, all but finitely many of them are contained in $\mathrm{ball}(\varepsilon, A)$. Hence $b \setminus C$ is covered with finitely many of the sets b_k, and so is $b \cap c_n \subseteq b \setminus C$. Since each b_k belongs to J, this contradicts the fact that $b \cap c_n \notin J$. □

Now, to finish the proof we will show that $J = J_I$. One inclusion is obvious: if $a \in J$, then $\mathrm{cl}(a) \in I$ by the definition of I and so $a \in J_I$. On the other hand, if $a \in J_I$, then $\mathrm{cl}(a) \in I$. Thus, there is $b \in J$ with $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$. We must prove that $a \in J$. However, if $a \notin J$, then for some n we have $c_n \cap b = \emptyset$ and $c_n \cap a \neq \emptyset$. Let $C \subseteq 2^\omega$ be a basic clopen set with $C \cap D = c_n$ and note that $\mathrm{cl}(b) \cap C = \emptyset$ and $\mathrm{cl}(a) \setminus C \neq \emptyset$, which contradicts $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$. Thus, it must be the case that $a \in J$, which concludes the argument that $J = J_I$ and ends the entire proof.

□

3. Representation via compact nowhere dense sets

In this section we prove Corollary 1.3.

Proof of Corollary 1.3. Suppose J is represented on X via a σ-ideal I. By Theorem 1.1, we can assume X is the Cantor space and J is a family of subsets of a countable dense set $D \subseteq 2^\omega$. Below we write d for the usual metric on the Cantor space.

Let U be the largest open set in I and let $C = 2^\omega \setminus U$. If $U = \emptyset$, then all sets in I are meager, so there is nothing to do. Thus, assume U is nonempty. Note that C is nonempty (we always assume that the ideals are proper) and has no isolated points since I contains all singletons. Therefore, C is a Cantor set. Find a sequence of nonempty clopen sets V_n such that $U = \bigcup_n V_n$ and $\mathrm{diam}(V_n) \to 0$. For each $n \in \omega$ find also a closed countable infinite set $D_n \subseteq C$ so that $D_n \cap D = \emptyset$, $D_n \cap D_m = \emptyset$ if $n \neq m$ and

$$(*)\qquad D_n \subseteq \mathrm{ball}(\inf_{x\in C, y\in V_n} d(x,y) + \frac{1}{n}, V_n).$$

Let $D' = \bigcup_n D_n$. Find a bijection $h : D \cap U \to D'$ such that for each n, $h \restriction D \cap V_n$ is a bijection from $D \cap V_n$ to D_n. Note that if $b \subseteq D \cap U$, then

$$\mathrm{cl}(b) \cap C \subseteq \mathrm{cl}(h(b)) \subseteq \mathrm{cl}(b) \cup D'.$$

Now, since $D' \in I$, we have

$$(**)\qquad \mathrm{cl}(b) \in I \quad \text{if and only if} \quad \mathrm{cl}(h(b)) \in I$$

Let $E = (D \cap C) \cup D'$ and note that E is dense in C. Indeed, first $D \cap C \setminus \mathrm{cl}(U)$ is dense in $C \setminus \mathrm{cl}(U)$, as $C \setminus \mathrm{cl}(U)$ is open, and second $\mathrm{cl}(D')$ contains $C \cap \mathrm{cl}(U)$ by $(*)$.

Consider now the bijection $\rho : D \to E$ given by $\rho \restriction D \cap C = \mathrm{id}$ and $\rho \restriction D \cap U = h$. Write J' for the ideal on E, which is the isomorphic copy of the ideal J via ρ. We claim that J' is represented on C by $I \restriction C = \{A \subseteq C : A \in I\}$. Note that the latter σ-ideal consists of sets which are meager on C.

It suffices to show that for $a \subseteq D \cap C$ and $b \subseteq D \cap U$ we have:

$$\mathrm{cl}(\rho(a)) = \mathrm{cl}(a) \in I \quad \text{if and only if} \quad a \in J'$$

$$\mathrm{cl}(\rho(b)) = \mathrm{cl}(h(b)) \in I \quad \text{if and only if} \quad b \in J'$$

The first equivalence follows from the assumption on I and J. The second one follows from $(**)$. This ends the proof.

□

4. Descriptive complexity of ideals with topological representations

Proof of Theorem 1.4. Suppose J is analytic and has a topological representation. By Theorem 1.1 there is a σ-ideal I of compact subsets of a Polish space X with a countable dense set D such that J_I is isomorphic to J. Consider the function $a \mapsto \mathrm{cl}(a)$ from $P(D)$ to $K(X)$ and note that it is Baire class 1. Since J_I is analytic, $I = \{A \in K(X) : \exists b \in J_I\ A \subseteq \mathrm{cl}(b)\}$ is also analytic, and hence $\mathbf{\Pi}^0_2$ by the theorem of Kechris–Louveau–Woodin [17, Theorem 11]. Therefore, J must be $\mathbf{\Pi}^0_3$ as a preimage of a $\mathbf{\Pi}^0_2$ set by a Baire class 1 function.

To check that J is in fact $\mathbf{\Pi}^0_3$-complete, we need the following standard fact.

Lemma 4.1. *All analytic ideals are $\mathbf{\Sigma}^0_2$-hard.*

Proof. This follows directly from the Jalali-Naini–Mathias–Talagrand theorem [2, Theorem 4.1.2]. Indeed, if J is analytic, then it has the Baire property and hence $\mathrm{Fin} \leq_{\mathrm{RB}} J$. From this we easily get a continuous reduction from $\mathrm{Fin} \subseteq 2^\omega$ (which is $\mathbf{\Sigma}^0_2$-complete) to J. □

We will now show that J_I is $\mathbf{\Pi}^0_3$-hard. The argument is based on ideas from [20]. Fix a point $x \in X$. Fix also a compatible metric on X and let $V_n = \mathrm{ball}(2^{-n}, x)$ for each $n \in \omega$. For each n define an ideal $J^n = \{a \cap V_n : a \in J_I\}$ on $D \cap V_n$. Note that each J^n is analytic, and hence $\mathbf{\Sigma}^0_2$-hard by Lemma 4.1. Therefore for each n there is $\phi_n : 2^\omega \to P(D \cap V_n)$ such that $\phi_n^{-1}(J^n) = \mathrm{Fin}$. Define $\phi : (P(\omega))^\omega \to P(D)$ by $\phi(\langle a_n : n \in \omega\rangle) = \bigcup_{n\in\omega} \phi_n(a_n)$. Let W be the set

$$\{\langle a_n : n \in \omega\rangle : \forall n \in \omega\ a_n \in \mathrm{Fin}\} = \{\langle a_n : n \in \omega\rangle : \forall n \in \omega\ \phi_n(a_n) \in J^n\}.$$

and note that W is $\mathbf{\Pi}^0_3$-complete. To finish the proof it suffices to show that $\phi^{-1}(J_I) = W$.

Suppose first that $\phi(\langle a_n : n \in \omega\rangle) \in J_I$, i.e. $\bigcup_{n\in\omega} \phi_n(a_n) \in J_I$. Then for each n we have $\phi_n(a_n) \in J_I$ and $\phi_n(a_n) \subseteq V_n$. Hence, for each n the set $\phi_n(a_n)$ is in J^n and $\langle a_n : n \in \omega\rangle \in W$.

On the other hand, if $\langle a_n : n \in \omega\rangle \in W$, i.e. for each n we have $\phi_n(a_n) \in J^n \subseteq J_I$, then $\mathrm{cl}(\phi_n(a_n)) \in I$. Since I is a σ-ideal, $\bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$ is in I. To prove that $\bigcup_{n\in\omega} \phi_n(a_n) \in J_I$ it suffices to show that $\mathrm{cl}(\bigcup_{n\in\omega} \phi_n(a_n)) \subseteq \bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$. Indeed, if $y \in \mathrm{cl}(\bigcup_{n\in\omega} \phi_n(a_n))$, then there is a sequence x_n of elements of $\bigcup_{n\in\omega} \phi_n(a_n)$ convergent to y. There are two cases:

- either there is m such that there are infinitely many x_n's in $\phi_m(a_m)$. In this case y is an element of $\mathrm{cl}(\phi_m(a_m))$;
- or in each of $\phi_m(a_m)$'s there are only finitely many x_n's. But then x_n's must converge to x and hence $y = x$.

In both cases we have $y \in \bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$, which ends the proof. □

Corollary 1.5 is now an adaptation of the proof of the Kechris–Louveau–Woodin theorem.

Proof of Corollary 1.5. Suppose $J = J_I$ for a σ-ideal I on a compact space X with a dense subset $D \subseteq X$ on which J lives. Consider the family $F = \{C \in K(X) : C \cap D \text{ is dense in } C\}$ and note that F is Borel (in fact, $\mathbf{\Pi}^0_3$) and the map $C \mapsto C \cap D$ is a Borel function from F to $P(D)$. Let $I' = I \cap F$ and note that I' is coanalytic as it is the preimage of J by the above function.

Case 1. Suppose I' can be separated from $F \setminus I'$ by a $\mathbf{\Pi}^0_2$ set. Then, since F is Borel, we get that I' is Borel. Hence, J is Borel as the preimage of I' via the Baire class 1 map $P(D) \ni a \mapsto \mathrm{cl}(a) \in F$. By Theorem 1.4 we get then that J is $\mathbf{\Pi}^0_3$-complete. In this case $I \cap K(X)$ is clearly analytic and hence it is $\mathbf{\Pi}^0_2$ by the Kechris–Louveau–Woodin theorem [16, Theorem 33.3].

Case 2. Suppose I' cannot be separated from $F \setminus I'$ by a $\mathbf{\Pi}^0_2$ set. Then, by the Kechris–Louveau–Woodin theorem [16, Theorem 21.22], there is a Cantor set C in $K(X)$ with $C \subseteq F = I' \cup (F \setminus I')$ such that $C \cap I'$ is countable dense in C. Write $Q = C \cap I'$. Consider the function $K \mapsto (\bigcup K) \cap D$ from $K(C)$ to $P(D)$. Since $\bigcup K$ is compact for a compact $K \subseteq K(C)$, we have that if $K \subseteq Q$, then $\bigcup K \in I'$ and hence $(\bigcup K) \cap D \in J_I$. On the other hand, if $K \not\subseteq Q$, then $\bigcup K$ does not belong to I but still belongs to F (since any compact union of sets in F belongs to F) and hence $(\bigcup K) \cap D \notin J$. This proves that the above function is a reduction from $K(Q)$ (which is a $\mathbf{\Pi}^1_1$-complete subset of $K(C)$) to J and shows $\mathbf{\Pi}^1_1$-completeness of J. Finally, in this case $I \cap K(X)$ cannot be $\mathbf{\Pi}^0_2$ and hence it is also $\mathbf{\Pi}^1_1$-complete by the Kechris–Louveau–Woodin theorem [16, Theorem 33.3]. □

5. Weakly selective ideals

We follow standard set-theoretic notation concerning trees. In particular, a *branch* through a tree $T \subseteq \omega^{<\omega}$ is a sequence $t \in \omega^\omega$ such that $t \restriction n \in T$ for every $n \in \omega$. The set of all branches through a tree T is denoted by $[T]$. Given a tree $T \subseteq \omega^{<\omega}$, we say that branch $(n_1, n_2, \ldots) \in [T]$ is *J-positive* if $\{n_1, n_2, \ldots\} \notin J$. Given a tree $T \subseteq \omega^{<\omega}$ and $t \in T$ we write $\mathrm{split}_T(t) = \{n \in \omega : t^\frown n \in T\}$. Given a family A of subsets of ω, we say that a tree $T \subseteq \omega^{<\omega}$ is *A-splitting* if for each $t \in T$ we have $\mathrm{split}_T(t) \in A$. Given a tree T, we call the sets $\mathrm{split}_T(t)$ for $t \in T$ the *splitting sets of* T.

A subclass of weakly selective ideals are the selective ideals (see Mathias [21], Farah [5] and Grigorieff [10]). An ideal J is *selective* [10, Definition 1.7] if every J-partition of ω admits a J-positive selector. Here, a *J-partition* of ω is a partition $\omega = \bigcup_n a_n$ with $\bigcup_{m>n} a_m \notin J$ for every n. Equivalently [5, Definition 1.1], J is selective if any sequence of J-positive sets a_n has a J-positive *diagonalization*, i.e. a set $a \notin J$ such that $a \setminus n \subseteq a_n$ for each n. Selective ideals have been studied by Grigorieff, who proved [10, Corollary 1.15] a characterization of selectivity in terms of branches of trees: an ideal J is selective if and only if every tree T with the property that any finite intersection of splitting sets of T is in J^+, has a J-positive branch. We will need the following

Lemma 5.1. *Let J be an ideal on ω. The following are equivalent:*

(a) *J is weakly selective,*
(b) *for each J-positive b, every co-$(J \restriction b)$-splitting tree has a J-positive branch.*

Proof. This follows from [10, Proposition 14]. □

The proof of Theorem 1.6 will be based on the above lemma as well as on some ideas of Hrušák from his Category Dichotomy [12, Theorem 5.20]. In particular, we will use a game $H(J)$, considered by Hrušák in [12] and Laflamme [19].

Proof of Theorem 1.6. (ii)⇒(i) Suppose $J = \bigcap_{l\in\Lambda} J_l$, where each J_l has a topological representation. Let $f : b \to \omega$ be a function with $b \notin J$. Suppose f is not constant on any J-positive subset of b. Pick $l \in \Lambda$ such that $b \notin J_l$. We will find a J_l-positive subset of b on which f is 1-1. Let X be a separable metric space, D its dense countable subset and I a σ-ideal of subsets of X such that J_l is isomorphic to J_I. Without loss of generality assume $b \subseteq D$. Write $B = \mathrm{cl}(b)$ and let $B' = B \setminus \bigcup\{U : U \text{ is basic open and } B \cap U \in I\}$. Note that B' is still I-positive. Enumerate all basic open sets in X which intersect B' into a sequence $\langle V_n : n < \omega\rangle$ and by induction on i construct a sequence of points $n_i \in b$ such that $f(n_i) \notin \{f(n_j) : j < i\}$ and $n_i \in V_i$. Once this is done put $b' = \{n_i : i < \omega\}$ and note that $b' \notin J_l$ since $\mathrm{cl}(b')$ contains B'. Obviously, then f is 1-1 on b'.

To perform the construction, suppose that points n_i are chosen for $i < k$ and let $a = b \cap V_k$. Note that $a \notin J_l$. By our assumption, f is not constant on any J_l-positive set, so it cannot assume finitely many values on a, which implies that there is $n_k \in a$ such that $f(n_k) \neq f(n_i)$ for all $i < k$. This ends the construction.

(i)⇒(ii) Suppose now that J is coanalytic ideal of subsets of ω. Let $D \subseteq [\omega]^\omega \times \omega^\omega$ be closed set such that $[\omega]^\omega \setminus J$ is the projection of D.

Consider the following game $H'(J)$. In his n-th turn Player II picks a set $a_n \in J$. Player I responds with a pair (k_n, m_n) with $k_n \in \omega \setminus a_n$ and $m_n \in n \cup \{\text{pass}\}$. Player I wins if at the end he has chosen infinitely many m_n's different than 'pass' and $(\bar{k}, \bar{m})$ belongs to D, where $\bar{k} = \{k_n : n < \omega\}$ and $\bar{m}$ is the sequence of those m_n's which are not equal to 'pass'.

The game $H'(J)$ is an unfolded version of the game $H(J)$ in which Player II picks $a_n \in J$ and Player I responds just with a number $k_n \notin a_n$. Player I wins in $H(J)$ if $\{k_n : n < \omega\}$ does not belong to J.

Claim 5.2. *If Player II has a winning strategy in $H'(J)$, then he also has a winning strategy in $H(J)$.*

Proof. Let σ be a winning strategy for Player II in $H'(J)$. We describe a strategy σ' for Player II in the game $H(J)$. Suppose Player II is about to make his n-th move after Player I has played $k_0, \ldots, k_{n-1}$. Let F be the finite set of all sequences $m_0, \ldots, m_n$ such that $m_i \in i \cup \{\text{pass}\}$ and for each $f = (m_0, \ldots, m_{n-1}) \in F$ let a_f be the n-th move in the game $H'(J)$ according to the strategy σ after Player I has played $(k_0, m_0), \ldots, (k_{n-1}, m_{n-1})$. Let the move of Player II in $H(J)$ be $\bigcup_{f \in F} a_f$.

We claim that this is a winning strategy for Player II. Suppose it is not and there is a counterplay of Player I. The counterplay is a sequence $(k_n : n < \omega)$ such that $\{k_n : n < \omega\} \notin J$. We will find a counterplay to the strategy σ in $H'(J)$. Since $\{k_n : n < \omega\} \notin J$, there is a sequence $(m_n : n < \omega) \in \omega^\omega$ such that $(\{k_n : n < \omega\}, (m_n : n < \omega)) \in D$. Let m'_n be the sequence such that $m_n \in i \cup \{\text{pass}\}$ and the elements of m'_n different from 'pass' enumerate $(m_n : n < \omega)$. Consider now the play in which Player I plays (k_n, m'_n) and Player II plays according to σ. Note that this is a legal play in $H'(J)$ by the definition of σ'. It is also a counterplay to σ in which Player I wins. $\square$

The following claim can be deduced from [19, Theorem 2.7] but we attach a short proof below.

Claim 5.3. *If J is weakly selective, then Player II cannot have a winning strategy in $H(J)$.*

Proof. Suppose there is such strategy and let T be the tree of all counterplays of Player I, i.e. $T = \{(k_0, \ldots, k_n) : n < \omega \text{ and } k_i \notin \sigma(k_0, \ldots, k_{i-1}) \text{ for each } i \leq k\}$. Note that T is a co-J-splitting tree whose all branches belong to J. But by Lemma 5.1, any co-J-splitting tree must have a J-positive branch. $\square$

Claim 5.4. *If J is weakly selective, then for every $b \notin J$ there is a countable family Y of J-positive subsets of b such that for every $a \in J$ there is $x \in Y$ with $x \cap a = \emptyset$.*

Proof. Given a set $b \notin J$ consider the ideal $J \restriction b = \{a \subseteq b : a \in J\}$ of subsets of b. Note that it is still weakly selective and coanalytic. Hence, by Claims 5.2 and 5.3 and the fact that the game $H'(J \restriction b)$ is a closed game, there is a winning strategy σ for Player I in $H'(J \restriction b)$. Let T be the tree of all partial plays in $H'(J \restriction b)$, i.e. all sequences $(a_0, n_0, m_0, a_1, n_1, m_1, \ldots, a_k, n_k, m_k)$ such that $(n_i, m_i) = \sigma((a_0, n_0, m_0, \ldots, a_i))$. Now, inductively, for each k find a subset T'_k of T consisting of sequences $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k)$ such that:

- $(a_0, n_0, m_0, \ldots, a_{k-1}, n_{k-1}, m_{k-1})$ belongs to T'_{k-1},
- if $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k)$ and $(a'_0, n_0, m_0, \ldots, a'_k, n_k, m_k)$ belong to T'_k, then $a_i = a'_i$ for each $i \leq k$.

Let $T' = \bigcup_k T'_k$ and put

$$S = \{(n_0, \ldots, n_k) \in \omega^{<\omega} : \exists (a_0, \ldots, a_k), (m_0, \ldots, m_k) \\ (a_0, n_0, m_0, \ldots, a_k, n_k, m_k) \in T'\}.$$

The tree S is a subtree of $b^{<\omega}$, whose all branches are $(J \restriction b)$-positive, as if $(n_0, n_1, \ldots) \in [S]$, then there are $(a_0, a_1, \ldots), (m_0, m_1, \ldots)$ such that $(a_0, n_0, m_0, a_1, n_1, m_1, \ldots) \in [T']$ and all branches through T' follow the strategy σ. The tree S is also $(J \restriction b)^+$-splitting. Indeed, if $a = \mathrm{split}_S(t) \in J$ for some $t = (n_0, \ldots, n_k) \in S$, then pick $(a_0, \ldots, a_k)$ and $(m_0, \ldots, m_k)$ such that $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k) \in T'_k$. Let then

$$(n, m) = \sigma((a_0, n_0, m_0 \ldots, a_k, n_k, m_k, a)).$$

Note that $n \notin a$ and by the construction, there is a' such that

$$(a_0, n_0, m_0 \ldots, a_k, n_k, m_k, a', n, m) \in T'_{k+1}.$$

So $(n_0, \ldots, n_k, n) \in S$ and this contradicts the fact that $n \notin a$.

Consider now the family of all splitting sets of S. We claim that this is the desired family. Indeed, if a intersects all these sets, then a contains a branch through S and therefore, it is J-positive. □

For each $b \notin J$ let Y_b be a countable family of subsets of b as in Claim 5.4. We say that a family X of J-positive sets is *almost separating* if for every $x \in X$ there are $x_0, x_1 \in X$ with $x_0, x_1 \subseteq x$ such that $x_0 \cap x_1 = \emptyset$ and $x \setminus (x_0 \cup x_1) \in J$. Note that if X is almost separating, then for every $x \in X$ and $n \in x$ there is $y \in X$ with $y \subseteq x$ and $n \notin y$.

We also say that a family X of J-positive sets is *almost closed under finite intersections* if for every $x_0, \ldots, x_n \in X$ if $\bigcap_{i \leq n} x_i \neq \emptyset$, then there is $y \in X$ such that $y \subseteq \bigcap_{i \leq n} x_i$ and $(\bigcap_{i \leq n} x_i) \setminus y \in J$.

Claim 5.5. *For each $b \notin J$ there is a countable family X of J-positive subsets of b which is almost closed under finite intersections, almost separating and such that for each $x \in X$ and $a \in J$ there is $y \in X$ with $y \subseteq x$ and $y \cap a = \emptyset$.*

Proof. First note that since $J \restriction x$ is not a maximal ideal (as it has the Baire property) for every J-positive set $x \subseteq b$, there are two disjoint complementary J-positive subsets of x, say $x(0), x(1)$. Let Z be a countable family of J-positive subsets of b such that $b \in Z$ and

- for every $c \in Z$ we have $Y_c \subseteq Z$,
- for every $c \in Z$ we have $c(0), c(1) \in Z$,
- for every $c_0, \ldots, c_n \in Z$ if $\bigcap_{i \leq n} c_i \notin J$, then $\bigcap_{i \leq n} c_i \in Z$.

Enumerate Z with infinite repetitions as $\{z_0, z_1, \ldots\}$. Now, by induction construct sets x_i as follows. Let $x_0 = z_0$ and $x_{i+1} = z_i \setminus \bigcup \{z_i \cap x_j : j < i \text{ and } z_i \cap x_j \in J\}$. Now, the family $X = \{x_i : i < \omega\}$ is as needed. Indeed, note that for each $i < \omega$ we have $x_i \subseteq z_i$ and $z_i \setminus x_i \in J$, so the properties of the family X follow immediately from the construction of the family Z. □

Given a set $b \notin J$ let X_b be a countable family of J-positive subsets of b as in Claim 5.5. Let

$$J_b = \{a \subseteq \omega : \forall x \in X_b \ \exists y \in X_b \quad y \subseteq x \ \wedge \ a \cap y = \emptyset\}.$$

Note that J_b is an ideal of subsets of ω.

Claim 5.6. *For each $b \notin J$ the ideal J is contained in J_b.*

Proof. This follows directly from the properties of X_b. □

Claim 5.7. *For each $b \notin J$ the ideal J_b has a topological representation.*

Proof. We will check that J_b is countably separated and dense.

J_b is countably separated by X_b. Indeed, Let $a \in J_b$ and $c \notin J_b$. Since $c \notin J_b$, there is $x \in X_b$ such that for no $y \in X_b$ with $y \subseteq x$ it is the case that $y \cap c = \emptyset$. Note that actually for each such y we have $y \cap c \notin J_b$. Now, since $a \in J_b$, there is $y \in X_b$ with $y \subseteq x$ and $y \cap a = \emptyset$.

To see that J_b is dense, let $c \subseteq \omega$ be infinite. We need to find an infinite $a \subseteq c$ such that $a \in J_b$. We can assume that $c \notin J_b$. This means that there is $x \in X_b$ such that for each y with $y \subseteq x$ we have $y \cap c \notin J_b$. Enumerate X_b as $\{x_i : i < \omega\}$. By induction on i, construct a strictly increasing sequence n_i and J-positive sets $y_i \in X_b$ such that

(a) $y_{i+1} \subseteq y_i$, $y_i \subseteq x$ (so $y_i \cap c \notin J_b$) and $n_i \in y_i \cap c$
(b) $x_i \setminus y_i$ contains an element of X_b.

We start with $y_{-1} = x$. To perform the induction step, use the fact that X_b is almost separating and find two J_b-positive sets $x_i(0), x_i(1) \in X_b$ which are subsets of x_i and such that $x_i \setminus (x_i(0) \cup x_i(1)) \in J$. If $y_{i-1} \cap x_i$ is empty, then put $y_i = y_{i-1}$ and pick any $n_i \in y_i \cap c$ bigger than n_{i-1}. If $y_{i-1} \cap x_i$ is nonempty, then it is J-positive. Note that at least one of $y_{i-1} \cap x_i(0)$ or $y_{i-1} \cap x_i(1)$ must be nonempty. Since X_b is almost closed under finite intersections, one of these sets contains an element of X_b, say y_i. Pick any $n_i \in y_i \cap c$ bigger than n_{i-1}. This ends the construction.

Put $a = \{n_i : i < \omega\}$. We claim that a belongs to J_b. Indeed, pick $x_i \in X_b$. By (b) $x_i \setminus y_i$ contains an element of X_b, say y. Hence, by (a), $a \cap y$ is finite. Since X_b is almost separating, we can further shrink y to $z \in X_b$ such that $z \cap a$ is empty. This shows that $a \in J_b$. □

Now, for each $b \notin J$ we have the ideal J_b such that $J \subseteq J_b$, $b \notin J_b$ and J_b has a topological representation. This implies that $J = \bigcap_{b \notin J} J_b$ is an intersection of ideals which have topological representations and ends the proof. □

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