Erdős-Ulam ideals vs. simple density ideals

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Introduction

One of the most classical examples of ideals is

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}$$

– the *ideal of asymptotic density zero sets*.

EU ideals and SD ideals

Just and Krawczyk [3] introduced a generalization of the above: if $f\colon \omega \to [0,\infty)$ is such that $\sum_{i=0}^{\infty} f(i) = \infty$ and $\frac{f(n)}{\sum_{i=0}^{n} f(i)} \to 0$, then

$$\mathcal{EU}_f = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i=0}^{n-1} f(i)} = 0 \right\}$$

is an *Erdős-Ulam ideal* $\mathcal{EU}_f$ (*EU ideal* in short).

Recently, in [1] the authors proposed another generalization of the ideal $\mathcal{Z}$: if $g\colon \omega \to [0,\infty)$ is such that $g(n) \to \infty$ and $\frac{n}{g(n)} \not\to 0$, then

$$\mathcal{Z}_g = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{g(n)} = 0 \right\}$$

is a *simple density ideal* (*SD ideal* in short).

Motivation

Definition 1 (AK, J. Tryba, [7])**.** An ideal $\mathcal{I}$ has property $(\star)$ provided that for any $A \subseteq \omega$, if $\mathcal{I} \restriction A$ is isomorphic to $\mathcal{I}$, then the witnessing isomorphism $f\colon \omega \to A$ is just an increasing enumeration of A.

Remark 1. By [7, Theorem 5.6], $\mathcal{Z}$ has property $(\star)$. However, this was the only known example.

Theorem 1 (AK, M. Popławski, J. Swaczyna, J. Tryba, [6])**.** *All SD ideals have property* $(\star)$.

How many SD ideals are there?

Proposition 1 (AK, [5])**.** *There are $\mathfrak{c}$ many non-isomorphic EU ideals which are not SD ideals.*

Proposition 2 (AK, [5])**.** *There are $\mathfrak{c}$ many non-isomorphic EU ideals which simultaneously are SD ideals.*

Theorem 2 (AK, M. Popławski, J. Swaczyna, J. Tryba, [6])**.** *There are $\mathfrak{c}$ many non-isomorphic SD ideals which are not EU ideals.*

When an SD ideal is an EU ideal?

Proposition 3 (AK, [5])**.** *An SD ideal $\mathcal{Z}_g$ is an EU ideal iff the sequence $(n/g(n))_{n\in\omega}$ is bounded.*

Theorem 3 (AK, [5])**.** *An SD ideal $\mathcal{I}$ is an EU ideal iff $\mathcal{Z} \subseteq \mathcal{I}$.*

Remark 2. Note that Theorem 3 does not involve the function generating simple density ideal.

When an EU ideal is an SD ideal?

Answering this question requires introducing some new notions...

A nice condition

Definition 2 (AK, [5])**.** We say that an ideal $\mathcal{I}$ is *right-shift-invariant* if for every $B \in \mathcal{I}$ and $C \subseteq \omega$ satisfying $|C \cap n| \leq |B \cap n|$ for all n, we have $C \in \mathcal{I}$.

Lemma 1 (AK, [5])**.** *If $\mathcal{I}$ is a right-shift-invariant EU ideal, then $\mathcal{Z} \subseteq \mathcal{I}$.*

Density ideals

We say that an ideal is a *density ideal* (in sense of Farah) if it is of the form

$$\mathrm{Exh}\left(\sup_{n\in\omega} \mu_n\right) = \left\{ A \subseteq \omega : \lim_{n\to\infty} \mu_n(A) = 0 \right\},$$

where (μ_n) is a sequence of measures on ω with finite and pairwise disjoint supports (cf. [2, Section 1.13.]).

The importance of density ideals in our studies is a consequence of [1, Theorem 3.2] and [2, Theorem 1.13.3.(a)]: EU ideals as well as SD ideals are density ideals.

A very, very bad condition

Definition 3 (AK, [5])**.** We say that $\mathrm{Exh}(\sup_{n\in\omega} \mu_n)$ is *almost uniformly distributed* if every B such that $|\{i \in B : \mu_n(\{i\}) < \frac{1}{|\mathrm{supp}(\mu_n)|}\}| = 0$ and for all $m \in \omega \setminus \{0\}$ there is $n_m \in \omega$ such that

$$\left| \left\{ i \in B : \frac{k}{|\mathrm{supp}(\mu_n)|} \leq \mu_n(\{i\}) < \frac{k+1}{|\mathrm{supp}(\mu_n)|} \right\} \right| \leq \frac{|\mathrm{supp}(\mu_n)|}{mk(k+1)}$$

for all $n > n_m$ and $k \in \omega \setminus \{0\}$, belongs to $\mathrm{Exh}(\sup_{n\in\omega} \mu_n)$.

Proposition 4 (AK, [5])**.** *If the sequence $(|\mathrm{supp}(\mu_n)|\mu_n(\{i\}))_{n,i\in\omega}$ is bounded, then the ideal $\mathrm{Exh}(\sup_{n\in\omega} \mu_n)$ is almost uniformly distributed.*

The characterization

Theorem 4 (AK, [5])**.** *An EU ideal is an SD ideal iff it is right-shift-invariant and almost uniformly distributed.*

Remark 3. We have examples showing that there is no implication between right-shift-invariance and almost uniform distribution.

Open problems

Problem 1. Which other ideals have property $(\star)$?

Problem 2. Is there a nice way of expressing almost uniform distribution (at least in case of EU ideals)?

Problem 3. Characterize density ideals which are SD ideals.

References

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