Rank of $\mathcal{F}$-limits of filter sequences

Adam Kwela

Institute of Mathematics, Polish Academy of Sciences

July 6, 2013

Based on joint paper with Ireneusz Recław, *Rank of $\mathcal{F}$-limits of filter sequences*, J. Math. Anal. Appl. 398 (2013), 872–878.

Rank of a filter

For a filter $\mathcal{F}$ on a countable set, by $\mathcal{F}^*$ we denote its dual ideal.

A is Γ-separated from B, if there exists a set $S \in \Gamma$ such that $A \subset S$ and $B \cap S = \emptyset$.

Definition (Debs–Saint Raymond, 2009)

Rank of an analytic filter $\mathcal{F}$ is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min \left\{ \alpha < \omega_1 : \mathcal{F} \text{ is } \Sigma^0_{1+\alpha}\text{-separated from } \mathcal{F}^* \right\}.$$

Rank of a filter

For a filter $\mathcal{F}$ on a countable set, by $\mathcal{F}^*$ we denote its dual ideal.

A is Γ-separated from B, if there exists a set $S \in \Gamma$ such that $A \subset S$ and $B \cap S = \emptyset$.

Definition (Debs–Saint Raymond, 2009)

Rank of an analytic filter $\mathcal{F}$ is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \text{ is } \Sigma^0_{1+\alpha}\text{-separated from } \mathcal{F}^*\right\}.$$

Rank of a filter

For a filter $\mathcal{F}$ on a countable set, by $\mathcal{F}^*$ we denote its dual ideal.

A is Γ-separated from B, if there exists a set $S \in \Gamma$ such that $A \subset S$ and $B \cap S = \emptyset$.

Definition (Debs–Saint Raymond, 2009)

Rank of an analytic filter $\mathcal{F}$ is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \text{ is } \Sigma^0_{1+\alpha}\text{-separated from } \mathcal{F}^*\right\}.$$

Notations

- By $\mathcal{B}_\alpha(X)$ we denote the family of all real-valued functions on X of Borel class α.
- Point $x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i \in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.
- By $\mathcal{C}_\mathcal{F}(X)$ we denote the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.
- By $\mathcal{F}_{Fr}$ we denote the Frechét filter, i.e., filter of all cofinite subsets of ω.
- $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr} = \{A \subset \omega \times \omega : \exists_n \forall_{m>n} A_m \in \mathcal{F}_{Fr}\}$, where A_m denotes the m-th vertical section of A.

Notations

- By $\mathcal{B}_\alpha(X)$ we denote the family of all real-valued functions on X of Borel class α.
- Point $x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i\in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.
- By $\mathcal{C}_\mathcal{F}(X)$ we denote the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.
- By $\mathcal{F}_{Fr}$ we denote the Frechét filter, i.e., filter of all cofinite subsets of ω.
- $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr} = \{A \subset \omega \times \omega : \exists_n \forall_{m>n} A_m \in \mathcal{F}_{Fr}\}$, where A_m denotes the m-th vertical section of A.

Notations

- By $\mathcal{B}_\alpha(X)$ we denote the family of all real-valued functions on X of Borel class α.
- Point $x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i\in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.
- By $\mathcal{C}_\mathcal{F}(X)$ we denote the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.
- By $\mathcal{F}_{Fr}$ we denote the Frechét filter, i.e., filter of all cofinite subsets of ω.
- $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr} = \{A \subset \omega \times \omega : \exists_n \forall_{m>n} A_m \in \mathcal{F}_{Fr}\}$, where A_m denotes the m-th vertical section of A.

Notations

- By $\mathcal{B}_\alpha(X)$ we denote the family of all real-valued functions on X of Borel class α.
- Point $x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i\in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.
- By $\mathcal{C}_{\mathcal{F}}(X)$ we denote the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.
- By $\mathcal{F}_{Fr}$ we denote the Frechét filter, i.e., filter of all cofinite subsets of ω.
- $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr} = \{A \subset \omega \times \omega : \exists_n \forall_{m>n} A_m \in \mathcal{F}_{Fr}\}$, where A_m denotes the m-th vertical section of A.

Notations

- By $\mathcal{B}_\alpha(X)$ we denote the family of all real-valued functions on X of Borel class α.
- Point $x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i\in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.
- By $\mathcal{C}_\mathcal{F}(X)$ we denote the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.

- By $\mathcal{F}_{Fr}$ we denote the Frechét filter, i.e., filter of all cofinite subsets of ω.
- $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr} = \{A \subset \omega \times \omega : \exists_n \forall_{m>n} A_m \in \mathcal{F}_{Fr}\}$, where A_m denotes the m-th vertical section of A.

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- *$\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.*
- *$\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.*

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ *for any Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \leq \alpha$.
- $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ *for any zero-dimensional Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \geq \alpha$.

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ *for any Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \leq \alpha$.
- $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ *for any zero-dimensional Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \geq \alpha$.

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{\mathsf{Fr}} \times \mathcal{F}_{\mathsf{Fr}}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- *$\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.*
- *$\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.*

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ *for any Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \leq \alpha$.
- $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ *for any zero-dimensional Polish space X if and only if* $\mathrm{rk}(\mathcal{F}) \geq \alpha$.

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{Fr} \times \mathcal{F}_{Fr}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- *$\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.*
- *$\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.*

Rank and filter limits

Theorem (Recław–Laczkovich, 2009)

Let $\mathcal{F}$ be a Borel filter and X be an uncountable Polish space. Then TFAE:

- $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$.
- $\mathrm{rk}(\mathcal{F}) = 1$.
- *$\mathcal{F}$ does not contain an isomorphic copy of $\mathcal{F}_{\mathsf{Fr}} \times \mathcal{F}_{\mathsf{Fr}}$.*

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then

- *$\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.*
- *$\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.*

If $\mathcal{F} \subset \mathcal{P}(I)$ is a filter on I, then by $\mathrm{dom}\,(\mathcal{F})$ we denote the domain of $\mathcal{F}$, i.e., the set I.

$\Sigma_{i \in I} X_i$ is the disjoint sum of $(X_i)_{i \in I}$.

Definition

For a filter $\mathcal{F}$ on I and a family of filters $(\mathcal{F}_i)_{i \in I}$, the collection of all sets of the form $\sum_{i \in F} F_i$, for $F \in \mathcal{F}$ and $F_i \in \mathcal{F}_i$, constitutes a basis of a filter on the set $\sum_{i \in I} \mathrm{dom}\,(\mathcal{F}_i)$.
We call it the $\mathcal{F}$-Fubini sum of the family $(\mathcal{F}_i)_{i \in I}$ and denote by $\mathcal{F} - \sum_{i \in I} \mathcal{F}_i$.

Fubini sums

If $\mathcal{F} \subset \mathcal{P}(I)$ is a filter on I, then by $\mathrm{dom}(\mathcal{F})$ we denote the domain of $\mathcal{F}$, i.e., the set I.

$\Sigma_{i \in I} X_i$ is the disjoint sum of $(X_i)_{i \in I}$.

Definition

For a filter $\mathcal{F}$ on I and a family of filters $(\mathcal{F}_i)_{i \in I}$, the collection of all sets of the form $\sum_{i \in F} F_i$, for $F \in \mathcal{F}$ and $F_i \in \mathcal{F}_i$, constitutes a basis of a filter on the set $\sum_{i \in I} \mathrm{dom}(\mathcal{F}_i)$.
We call it the $\mathcal{F}$-Fubini sum of the family $(\mathcal{F}_i)_{i \in I}$ and denote by $\mathcal{F} - \sum_{i \in I} \mathcal{F}_i$.

Fubini sums

If $\mathcal{F} \subset \mathcal{P}(I)$ is a filter on I, then by $\operatorname{dom}(\mathcal{F})$ we denote the domain of $\mathcal{F}$, i.e., the set I.

$\Sigma_{i\in I} X_i$ is the disjoint sum of $(X_i)_{i\in I}$.

Definition

For a filter $\mathcal{F}$ on I and a family of filters $(\mathcal{F}_i)_{i\in I}$, the collection of all sets of the form $\sum_{i\in F} F_i$, for $F \in \mathcal{F}$ and $F_i \in \mathcal{F}_i$, constitutes a basis of a filter on the set $\sum_{i\in I} \operatorname{dom}(\mathcal{F}_i)$.
We call it the $\mathcal{F}$-Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$ and denote by $\mathcal{F} - \sum_{i\in I} \mathcal{F}_i$.

Rank of Fubini sums

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \in \mathcal{F}$.

- *If $\mathrm{rk}(\mathcal{F}) \geq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.*
- *If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.*
- *If $\mathcal{F} = \mathcal{F}_{Fr}$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1$.*

Question

What is the exact value for $\mathcal{F} \neq \mathcal{F}_{Fr}$?

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \in \mathcal{F}$.

- *If* $\mathrm{rk}(\mathcal{F}) \geq \alpha$ *and* $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.
- *If* $\mathrm{rk}(\mathcal{F}) \leq \alpha$ *and* $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.
- *If* $\mathcal{F} = \mathcal{F}_{Fr}$ *and* $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \leq \beta + 1$.

Question

What is the exact value for $\mathcal{F} \neq \mathcal{F}_{Fr}$*?*

Rank of Fubini sums

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \in \mathcal{F}$.

- *If $\mathrm{rk}(\mathcal{F}) \geq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.*
- *If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.*
- *If $\mathcal{F} = \mathcal{F}_{Fr}$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1$.*

Question

What is the exact value for $\mathcal{F} \neq \mathcal{F}_{Fr}$?

Rank of Fubini sums

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \in \mathcal{F}$.

- *If* $\mathrm{rk}(\mathcal{F}) \geq \alpha$ *and* $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.
- *If* $\mathrm{rk}(\mathcal{F}) \leq \alpha$ *and* $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.
- *If* $\mathcal{F} = \mathcal{F}_{Fr}$ *and* $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ *for all* $i \in J$*, then* $\mathrm{rk}(\mathcal{G}) \leq \beta + 1$.

Question

What is the exact value for $\mathcal{F} \neq \mathcal{F}_{Fr}$*?*

Rank of Fubini sums

Theorem (Debs–Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \in \mathcal{F}$.

- *If $\mathrm{rk}(\mathcal{F}) \geq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.*
- *If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.*
- *If $\mathcal{F} = \mathcal{F}_{Fr}$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1$.*

Question

What is the exact value for $\mathcal{F} \neq \mathcal{F}_{Fr}$?

Definition

If $\mathcal{F}$ is a filter on I and $(\mathcal{F}_i)_{i\in I}$ are filters on X, then

$$\lim_{\mathcal{F}} \mathcal{F}_i = \{A \subset X : \{i \in I : A \in \mathcal{F}_i\} \in \mathcal{F}\}$$

is a filter on X called the $\mathcal{F}$-limit of filters $(\mathcal{F}_i)_{i\in I}$.

Question

Is there a similar theorem to the previous one for $\mathcal{F}$-limits?

$\mathcal{F}$-limits of filter sequences

Definition

If $\mathcal{F}$ is a filter on I and $(\mathcal{F}_i)_{i\in I}$ are filters on X, then

$$\lim_{\mathcal{F}} \mathcal{F}_i = \{A \subset X : \{i \in I : A \in \mathcal{F}_i\} \in \mathcal{F}\}$$

is a filter on X called the $\mathcal{F}$-limit of filters $(\mathcal{F}_i)_{i\in I}$.

Question

Is there a similar theorem to the previous one for $\mathcal{F}$-limits?

Proposition (K–Recław)

Let $J \subset I$ be an element of $\mathcal{F}$. If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1 + \alpha$.

Theorem (K–Recław)

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1$.

Corollary

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) = \beta$ for all $i \in J$, then $\mathrm{rk}\left(\mathcal{F} - \sum_{i \in I} \mathcal{F}_i\right) = \beta + 1$.

Rank of $\mathcal{F}$-limits

Proposition (K–Recław)

Let $J \subset I$ be an element of $\mathcal{F}$. If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1 + \alpha$.

Theorem (K–Recław)

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1$.

Corollary

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) = \beta$ for all $i \in J$, then $\mathrm{rk}\left(\mathcal{F} - \sum_{i \in I} \mathcal{F}_i\right) = \beta + 1$.

Rank of $\mathcal{F}$-limits

Proposition (K–Recław)

Let $J \subset I$ be an element of $\mathcal{F}$. If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1 + \alpha$.

Theorem (K–Recław)

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1$.

Corollary

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) = \beta$ for all $i \in J$, then $\mathrm{rk}\left(\mathcal{F} - \sum_{i \in I} \mathcal{F}_i\right) = \beta + 1$.

Method of Recław and Laczkovich

Consider the following game $G(\mathcal{F})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{F}$ and Player II responds with a finite set $F_n \subset C_n$. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{F}^*$.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{F})$ is a determined game.
2. Laczkovich–Recław: Player I has a winning strategy in $G(\mathcal{F})$ if and only if $\mathrm{rk}(\mathcal{F}) > 1$
3. Therefore Player II has a winning strategy in $G(\mathcal{F})$.
4. Laflamme's characterization of $G(\mathcal{F})$: $\mathcal{F}$ has some "nice" combinatorial properties.

Method

Method of Recław and Laczkovich

Consider the following game $G(\mathcal{F})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{F}$ and Player II responds with a finite set $F_n \subset C_n$. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{F}^*$.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{F})$ is a determined game.
2. Laczkovich–Recław: Player I has a winning strategy in $G(\mathcal{F})$ if and only if $\mathrm{rk}(\mathcal{F}) > 1$
3. Therefore Player II has a winning strategy in $G(\mathcal{F})$.
4. Laflamme's characterization of $G(\mathcal{F})$: $\mathcal{F}$ has some "nice" combinatorial properties.

Method

Method of Recław and Laczkovich

Consider the following game $G(\mathcal{F})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{F}$ and Player II responds with a finite set $F_n \subset C_n$. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{F}^*$.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{F})$ is a determined game.
2. Laczkovich–Recław: Player I has a winning strategy in $G(\mathcal{F})$ if and only if $\mathrm{rk}(\mathcal{F}) > 1$
3. Therefore Player II has a winning strategy in $G(\mathcal{F})$.
4. Laflamme's characterization of $G(\mathcal{F})$: $\mathcal{F}$ has some "nice" combinatorial properties.

Method

Method of Recław and Laczkovich

Consider the following game $G(\mathcal{F})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{F}$ and Player II responds with a finite set $F_n \subset C_n$. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{F}^*$.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{F})$ is a determined game.
2. Laczkovich–Recław: Player I has a winning strategy in $G(\mathcal{F})$ if and only if $\mathrm{rk}(\mathcal{F}) > 1$
3. Therefore Player II has a winning strategy in $G(\mathcal{F})$.
4. Laflamme's characterization of $G(\mathcal{F})$: $\mathcal{F}$ has some "nice" combinatorial properties.

Method of Recław and Laczkovich

Consider the following game $G(\mathcal{F})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{F}$ and Player II responds with a finite set $F_n \subset C_n$. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{F}^*$.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{F})$ is a determined game.
2. Laczkovich–Recław: Player I has a winning strategy in $G(\mathcal{F})$ if and only if $\mathrm{rk}(\mathcal{F}) > 1$
3. Therefore Player II has a winning strategy in $G(\mathcal{F})$.
4. Laflamme's characterization of $G(\mathcal{F})$: $\mathcal{F}$ has some "nice" combinatorial properties.

Bottom estimate of rank of $\mathcal{F}$-limits

Theorem (K–Recław)

For any ordinals $\alpha, \beta < \omega_1$, there are $\mathcal{F}$ of rank β and $(\mathcal{F}_i)_{i \in \mathrm{dom}(\mathcal{F})}$ of ranks α such that $\lim_{\mathcal{F}} \mathcal{F}_i$ has rank 1.

Lemma (K–Recław)

For any $\alpha < \omega_1$, there exist two filters $\mathcal{G}_0$ and $\mathcal{G}_1$ of rank α and such that $\mathcal{G}_0 \cap \mathcal{G}_1$ has rank 1.

Bottom estimate of rank of $\mathcal{F}$-limits

Theorem (K–Recław)

For any ordinals $\alpha, \beta < \omega_1$, there are $\mathcal{F}$ of rank β and $(\mathcal{F}_i)_{i \in \mathrm{dom}(\mathcal{F})}$ of ranks α such that $\lim_{\mathcal{F}} \mathcal{F}_i$ has rank 1.

Lemma (K–Recław)

For any $\alpha < \omega_1$, there exist two filters $\mathcal{G}_0$ and $\mathcal{G}_1$ of rank α and such that $\mathcal{G}_0 \cap \mathcal{G}_1$ has rank 1.

Thank you for your attention!

A.Kwela@impan.pl