Rank of $\mathcal{F}$-limits of filter sequences

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Rank of a filter

For a filter $\mathcal{F} \subset \mathcal{P}(I)$ on a countable set I, by $\mathcal{F}^*$ we denote its dual ideal. $\operatorname{dom}(\mathcal{F})$ is the domain of $\mathcal{F}$, i.e., I. $\mathcal{F}_{Fr}$ is the Frechét filter (filter of all cofinite subsets of I).

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Definition (Debs and Saint Raymond, 2009)

Rank of an analytic filter $\mathcal{F}$ is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \textit{ is } \Sigma^0_{1+\alpha}\textit{-separated from } \mathcal{F}^*\right\}.$$

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Theorem (Debs and Saint Raymond, 2009)

Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ be a countable ordinal. Then
(a) $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.
(b) $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.

$x \in X$ is a limit relatively to $\mathcal{F}$ of a sequence $(x_i)_{i\in I} \subset X$, if $\{i \in I : \rho_X(x_i, x) < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.

$\mathcal{C}_{\mathcal{F}}(X)$ is the family of all real-valued functions on X, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of a sequence of continuous functions.

$\mathcal{B}_\alpha(X)$ is the family of all real-valued functions on X of Borel class α.

Rank of Fubini sums

$\Sigma_{i\in I}X_i$ is the disjoint sum of $(X_i)_{i\in I}$.

Definition

For a filter $\mathcal{F}$ on I and a family of filters $(\mathcal{F}_i)_{i\in I}$, the collection of all sets of the form $\sum_{i\in F} F_i$, for $F\in\mathcal{F}$ and $F_i\in\mathcal{F}_i$, constitutes a basis of a filter on the set $\sum_{i\in I}\mathrm{dom}(\mathcal{F}_i)$. We call it the $\mathcal{F}$-Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$ and denote it by $\mathcal{F}-\sum_{i\in I}\mathcal{F}_i$.

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Proposition (Debs and Saint Raymond, 2009)

Let $\mathcal{G}$ be the $\mathcal{F}$-Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $J \subset I$ be an element of $\mathcal{F}$.
(a) If $\mathrm{rk}(\mathcal{F}) \geq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \geq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \geq \beta + \alpha$.

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(b) If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \beta + 1 + \alpha$.

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(b) If $\mathrm{rk}(\mathcal{F})\leq\alpha$ and $\mathrm{rk}(\mathcal{F}_i)\leq\beta$ for all $i\in J$, then $\mathrm{rk}(\mathcal{G})\leq\beta+1+\alpha$.

(c) If $\mathcal{F}=\mathcal{F}_{Fr}$ and $\mathrm{rk}(\mathcal{F}_i)\leq\beta$ for all $i\in J$, then $\mathrm{rk}(\mathcal{G})\leq\beta+1$.

Definition

If $\mathcal{F}$ is a filter on I and $(\mathcal{F}_i)_{i \in I}$ are filters on X, then

$$\lim_{\mathcal{F}} \mathcal{F}_i = \{A \subset X : \{i \in I : A \in \mathcal{F}_i\} \in \mathcal{F}\}$$

is a filter on X called the $\mathcal{F}$-limit of filters $(\mathcal{F}_i)_{i \in I}$.

Rank of $\mathcal{F}$-limits

Proposition

Let $J \subset I$ be an element of $\mathcal{F}$. If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1 + \alpha$.

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Theorem (K. and Recław)

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1$.

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Corollary

Let $\mathcal{F}$ be a Borel filter on I of rank 1 and $J \in \mathcal{F}$. If $\mathrm{rk}(\mathcal{F}_i) = \beta$ for all $i \in J$, then $\mathrm{rk}\left(\mathcal{F} - \sum_{i \in I} \mathcal{F}_i\right) = \beta + 1$.

Sketch of the proof

For a filter $\mathcal{F}$, a set $Z = \left\{Z^k : k \in \omega\right\} \subset Fin \setminus \{\emptyset\}$ is called $\mathcal{F}$-universal, if for every element $M \in \mathcal{F}$, there is $k \in \omega$ such that $Z^k \subset M$.

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$\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets, if there are $\mathcal{F}$-universal sets $Z_n = \left\{Z_n^k : k \in \omega\right\}$ such that for every $M \in \mathcal{F}$ there is n such that for all but finitely many k $Z_n^k \cap M \neq \emptyset$.

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Lemma (Laczkovich and Recław, 2009)

If $\mathcal{F}$ is a Borel filter of rank 1, then there is a family $\{Z_n^k : k \in \omega\}_{n \in \omega}$ of $\mathcal{F}$-universal sets ω-diagonalizing $\mathcal{F}$.

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Let $S_i \in \Pi^0_{1+\beta}(X)$ be sets separating the filters $\mathcal{F}_i$ from their dual ideals. Define

$$S = \bigcup_{n \in \omega} \bigcup_{m \in \omega} \bigcap_{k > m} \bigcup_{i \in Z_n^k \cap J} S_i.$$

S is of additive class $1 + \beta + 1$ and separates $\lim_{\mathcal{F}} \mathcal{F}_i$ from its dual ideal.

Theorem (K. and Recław)

For every ordinals $\alpha, \beta < \omega_1$, there are $\mathcal{F}$ of rank β and $(\mathcal{F}_i)_{i \in \mathrm{dom}(\mathcal{F})}$ of ranks α such that $\lim_{\mathcal{F}} \mathcal{F}_i$ has rank 1.

Bottom estimate of rank of $\mathcal{F}$-limits

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For every $\alpha < \omega_1$, there exist two filters $\mathcal{G}_0$ and $\mathcal{G}_1$ of rank α and such that $\mathcal{G}_0 \cap \mathcal{G}_1$ has rank 1.

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Take any (non-maximal) filter $\mathcal{F}$ of rank β and a set H such that $H \notin \mathcal{F}$ and $\mathrm{dom}(\mathcal{F}) \setminus H \notin \mathcal{F}$. Set $\mathcal{F}_i = \mathcal{G}_0$ for $i \in H$ and $\mathcal{F}_i = \mathcal{G}_1$ for $i \notin H$. Then $\lim_{\mathcal{F}} \mathcal{F}_i$ is equal to $\mathcal{G}_0 \cap \mathcal{G}_1$.

Thank you for your attention!

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