Ideal equal Baire classes

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Three notions of ideal convergence

Let $\mathcal{I}, \mathcal{J}$ be ideals on ω and let X be a set. Suppose that $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$, $(f_n) \subseteq \mathbb{R}^X$ and $f \in \mathbb{R}^X$.

- $x_n \xrightarrow{\mathcal{I}} x$ if $\{n : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$.
- $f_n \xrightarrow{\mathcal{I}} f$ (*$\mathcal{I}$-pointwise convergence*) if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for all $x \in X$;
- $f_n \xrightarrow{\mathcal{I}-d} f$ (*$\mathcal{I}$-discrete convergence*) if $\{n : f_n(x) \neq f(x)\} \in \mathcal{I}$ for all $x \in X$;
- $f_n \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ (*$(\mathcal{I}, \mathcal{J})$-equal convergence*) if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for all $x \in X$.

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A function $f\colon X \to \mathbb{R}$ is *quasi-continuous in* $x_0 \in X$ if for every $\varepsilon > 0$ and every open neighborhood U of x_0 there exists an open nonempty $V \subseteq U$ such that for all $x \in V$ we have $|f(x) - f(x_0)| < \varepsilon$. f is *quasi-continuous* ($f \in QC(X)$) if it is quasi-continuous in all $x \in X$.

Plan of the talk

	$\mathcal{I}$-pointwise convergence	$\mathcal{I}$-discrete convergence	$(\mathcal{I}, \mathcal{J})$-equal convergence
$C(X)$ X – perfectly normal space	✓	✓	?
$QC(X)$ X – metric Baire space	✓	✓	?

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QC(X) + $\mathcal{I}$-pointwise convergence

Theorem (Z. Grande)

1. $B_1(QC(X)) = PWD(X)$ *for all metric Baire spaces* X.
2. $B_\alpha(QC(X)) = Baire(X)$ *for all* $\alpha > 1$ *and all metric Baire spaces* X.

Theorem (P. Szuca and T. Natkaniec; A.K.)

Let $\mathcal{I}$ *be a Borel ideal.*

1. $\mathcal{WR} \not\sqsubseteq \mathcal{I}$ *if and only if* $B_1^{\mathcal{I}}(QC(X)) = PWD(X)$ *for all metric Baire spaces* X.
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QC(X) + $\mathcal{I}$-discrete convergence

Theorem (Z. Grande)

1. $B_1^d(QC(X)) = PWD_0(X)$ *for all metric Baire spaces* X.
2. $B_\alpha^d(QC(X)) = Baire(X)$ *for all* $\alpha > 1$ *and all metric Baire spaces* X.

Theorem (P. Szuca and T. Natkaniec; A.K.)

Let $\mathcal{I}$ *be a Borel ideal.*

1. $\mathcal{WR} \not\sqsubseteq \mathcal{I}$ *if and only if* $B_1^{\mathcal{I}-d}(QC(X)) = PWD_0(X)$ *for all metric Baire spaces* X.
2. $\mathcal{WR} \sqsubseteq \mathcal{I}$ *if and only if* $B_1^{\mathcal{I}-d}(QC(X)) = Baire(X)$ *for all metric Baire spaces* X.
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$\mathcal{J}$ *contains an isomorphic copy of* $\mathcal{I}$ ($\mathcal{I} \sqsubseteq \mathcal{J}$) if there is a bijection $f\colon \bigcup\mathcal{I} \to \bigcup\mathcal{J}$ such that $f[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$.

$\mathcal{WR}$ is an ideal on $\omega \times \omega$ generated by two kinds of generators:

1. vertical lines, i.e., sets of the form $\{n\} \times \omega$ for $n \in \omega$;
2. sets $G \subseteq \omega \times \omega$ such that for every $(i,j),(k,l) \in G$ either $i > k + l$ or $k > i + j$.

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QC(X) + $(\mathcal{I}, \mathcal{J})$-equal convergence

Theorem (folklore)

1. $B^e_1(QC(X)) = PWD_0(X)$ *for all metric Baire spaces* X.
2. $B^e_\alpha(QC(X)) = Baire(X)$ *for all* $\alpha > 1$ *and all metric Baire spaces* X.

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Theorem (A.K. and M. Staniszewski)

Let $\mathcal{I}$ and $\mathcal{J}$ be non-orthogonal ideals on ω. Suppose that $\mathcal{I}$ is Borel.

1. *$(\mathcal{I}, \mathcal{J})$ is of the first q-type if and only if $B_1^{(\mathcal{I},\mathcal{J})-e}(QC(X)) = PWD_0(X)$ for all metric Baire X.*
2. *$(\mathcal{I}, \mathcal{J})$ is of the second q-type if and only if $B_1^{(\mathcal{I},\mathcal{J})-e}(QC(X)) = PWD(X)$ for all metric Baire X.*
3. *$(\mathcal{I}, \mathcal{J})$ is of the third q-type if and only if $B_1^{(\mathcal{I},\mathcal{J})-e}(QC(X)) = Baire(X)$ for all metric Baire X.*
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Theorem (A.K. and M. Staniszewski)

Let $\mathcal{I}$ and $\mathcal{J}$ be non-orthogonal ideals on ω. Suppose that $\mathcal{I}$ is Borel.

1. $(\mathcal{I},\mathcal{J})$ *is of the first q-type if and only if* $B_1^{(\mathcal{I},\mathcal{J})-e}(QC(X)) = PWD_0(X)$ *for all metric Baire* X.
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q-types

Let $A \in \mathcal{P}(\omega)$ and $(A_n)_{n\in\omega} \subseteq \mathcal{P}(\omega)$. We denote:

- $\mathcal{I} \sqcup A = \{M \cup N : \ M \in \mathcal{I} \wedge N \subseteq A\}$;
- $\mathcal{I} \sqcup (A_n)_{n\in\omega} = \left\{M \cup N : \ M \in \mathcal{I} \ \wedge \ \exists_{n\in\omega} \ N \subseteq \bigcup_{i<n} A_i\right\}$.

Definition

1. $(\mathcal{I}, \mathcal{J})$ *is of the first q-type if* $\mathcal{WR} \not\sqsubseteq \mathcal{I} \sqcup (A_n)_{n\in\omega}$ *for any sequence* $(A_n)_{n\in\omega} \subseteq \mathcal{J}$.
2. $(\mathcal{I}, \mathcal{J})$ *is of the second q-type if* $\mathcal{WR} \sqsubseteq \mathcal{I} \sqcup (A_n)_{n\in\omega}$ *for some sequence* $(A_n)_{n\in\omega} \subseteq \mathcal{J}$*, but* $\mathcal{WR} \not\sqsubseteq \mathcal{I} \sqcup A$ *for any* $A \in \mathcal{J}$.
3. $(\mathcal{I}, \mathcal{J})$ *is of the third q-type if* $\mathcal{WR} \sqsubseteq \mathcal{I} \sqcup A$ *for some* $A \in \mathcal{J}$.

Example

1. $(\mathrm{Fin}, \mathrm{Fin} \otimes \emptyset)$ is of the first q-type.
2. $(\emptyset \otimes \mathrm{Fin}, \mathrm{Fin} \otimes \emptyset)$ is of the second q-type.
3. $(\mathcal{WR}, \mathrm{Fin} \otimes \emptyset)$ is of the third q-type.

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Method of Laczkovich and Recław

Consider the following game $G_1(\mathcal{I})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{I}$ and Player II responds with a $k_n \notin C_n$. Player I wins, if $\bigcup_{n\in\omega}\{k_n\} \in \mathcal{I}$. Otherwise Player II wins.

1. Martin's Theorem on Borel determinacy: $G_1(\mathcal{I})$ is determined.
2. K.: Player I has a winning strategy if and only if $\mathcal{WR} \sqsubseteq \mathcal{I}$
3. Laflamme: Player II has a winning strategy if and only if $\mathcal{I}$ is ω-+-diagonalizable: there is a sequence $(D_n)_{n\in\omega}$ such that for each $A \in \mathcal{I}$ there is n with $A \cap D_n = \emptyset$.

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Definition

1. $(\mathcal{I},\mathcal{J})$ *is of the first c-type if* $Fin\otimes Fin\not\sqsubseteq\mathcal{I}\sqcup(A_n)_{n\in\omega}$ *for any sequence* $(A_n)_{n\in\omega}\subseteq\mathcal{J}$.
2. $(\mathcal{I},\mathcal{J})$ *is of the second c-type if* $Fin\otimes Fin\sqsubseteq\mathcal{I}\sqcup(A_n)_{n\in\omega}$ *for some sequence* $(A_n)_{n\in\omega}\subseteq\mathcal{J}$, *but* $Fin\otimes Fin\not\sqsubseteq\mathcal{I}\sqcup A$ *for any* $A\in\mathcal{J}$.
3. $(\mathcal{I},\mathcal{J})$ *is of the third c-type if* $Fin\otimes Fin\sqsubseteq\mathcal{I}\sqcup A$ *for some* $A\in\mathcal{J}$.

Method of Laczkovich and Recław

Consider the following game $G_2(\mathcal{I})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{I}$ and Player II responds with a finite set F_n disjoint with C_n. Player I wins, if $\bigcup_{n\in\omega} F_n \in \mathcal{I}$. Otherwise Player II wins.

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C(X) + $(\mathcal{I},\mathcal{J})$-equal convergence

Theorem (A.K. and M. Staniszewski)

Let $\mathcal{I}$ and $\mathcal{J}$ be non-orthogonal ideals on ω. Suppose that $\mathcal{I}$ is Borel.

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Generalize the previous Theorem to all $1 \leq \alpha < \omega_1$.

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Characterize $B_1^{(\mathcal{I},\mathcal{J})-e}(C(X))$ *for* $(\mathcal{I},\mathcal{J})$ *of the third c-type.*

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Thank you for your attention!