Ideal weak QN-spaces

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Ideal version of $\mathfrak{b}$

$\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal if it is closed under subsets and finite unions, contains $\mathrm{Fin} = [\omega]^{<\omega}$ and $\omega \notin \mathcal{I}$.

Let $\mathcal{P}_\mathcal{I}$ denote the family of all partitions of ω into sets from $\mathcal{I}$. $\mathcal{I}$ is a weak P-ideal if for each $(A_n) \in \mathcal{P}_\mathcal{I}$ we can find $M \notin \mathcal{I}$ with $M \cap A_n$ finite for each n.

$\mathtt{non}(\mathcal{I}\text{QN-space})$ ($\mathtt{non}(\mathcal{I}\text{wQN-space})$) denotes the minimal cardinality of a perfectly normal space which is not $\mathcal{I}$QN ($\mathcal{I}$wQN).

Theorem (Filipów and Staniszewski; Šupina)

$\mathtt{non}(\mathcal{I}\textit{QN-space}) =$

$$\min\left\{|\mathcal{A}| : \ \mathcal{A} \subseteq Fin^{\omega} \ \wedge \ \forall_{(D_n)\in\mathcal{P}_\mathcal{I}} \exists_{(A_n)\in\mathcal{A}} \bigcup_{n\in\omega} A_n \cap D_n \notin \mathcal{I}\right\}.$$

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where $e_B:\omega\to B$ is an increasing enumeration of B.

Theorem (Bukovský, Recław and Repický)

$\mathrm{non}(Fin\text{QN-space}) = \mathrm{non}(Fin\,w\text{QN-space}) = \mathfrak{b}$.

Theorem

$\mathfrak{b}\leq\mathrm{non}(\mathcal{I}\text{QN-space})\leq\mathrm{non}(\mathcal{I}w\text{QN-space})\leq\mathfrak{d}$ *for all weak P-ideals.*

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Nice ideals

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$\mathrm{non}(\mathcal{I}QN\text{-space}) = \mathrm{non}(\mathcal{I}wQN\text{-space}) = \mathfrak{b}$ for all $\mathtt{F}_\sigma$ ideals.

Corollary

$\mathrm{non}(\mathcal{I}QN\text{-space}) = \mathrm{non}(\mathcal{I}wQN\text{-space}) = \mathfrak{b}$ for every ideal contained in some $\mathtt{F}_\sigma$ ideal.

By a result of Solecki, each analytic P-ideal is of the form Exh(ϕ) for some lower semi-continuous submeasure ϕ. Fin(ϕ) is $\mathtt{F}_\sigma$ and we have Exh(ϕ) $\subseteq$ Fin(ϕ). If $\phi(\omega) = \infty$, then Fin(ϕ) becomes an ideal and we obtain $\mathrm{non}(\text{Exh}(\phi)\text{QN-space}) = \mathfrak{b}$.

This class contains all density ideals (in the sense of Farah), which are not EU.

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Bad ideals

Theorem

If $\mathfrak{b} < \mathfrak{b}_{\mathcal{J}}$ for some $\mathcal{J}$, then there is a weak P-ideal $\mathcal{I}$ with $\mathtt{non}(\mathcal{I}wQN\text{-space}) > \mathfrak{b}$.

Proof.

Show that

$$\mathcal{I} = (\mathrm{Fin} \otimes \mathrm{Fin}) \cap (\emptyset \otimes \mathcal{J})$$

is a weak P-ideal and $\mathtt{non}(\mathcal{I}\mathrm{wQN\text{-}space}) \geq \mathfrak{b}_{\mathcal{J}}$. □

Theorem (Canjar)

There is a maximal ideal $\mathcal{J}$ with $\mathfrak{b}_{\mathcal{J}} = \mathtt{cf}(\mathfrak{d})$.

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Ideal convergence

Suppose that $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$, $(f_n) \subseteq \mathbb{R}^X$ and $f \in \mathbb{R}^X$.

- $x_n \xrightarrow{\mathcal{I}} x$ if $\{n : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$;
- $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ (*$\mathcal{I}$-quasi-normal convergence*) if there exists a sequence of positive reals $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for all $x \in X$.

FinQN convergence is the σ-uniform convergence.

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- X is QN if any sequence $(f_n) \subseteq \mathbb{R}^X$ of continuous functions converging to zero FinQN converges to zero.
- X is wQN if for any sequence $(f_n) \subseteq \mathbb{R}^X$ of continuous functions converging to zero there is a subsequence (f_{n_k}) FinQN converging to zero.
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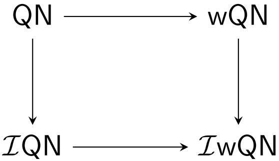

Theorem (Šupina)

For a non-weak P-ideal every topological space is $\mathcal{I}$QN and $\mathcal{I}$wQN.

Theorem (Šupina)

If $\mathfrak{p} = \mathfrak{c}$, then there is a weak P-ideal $\mathcal{I}$ and an $\mathcal{I}$QN but not QN-space.

This space is wQN, so we still need to distinguish wQN and $\mathcal{I}$wQN.

Theorem

If $\mathfrak{b} < \mathfrak{b}_{\mathcal{J}}$ for some $\mathcal{J}$, then there are a weak P-ideal $\mathcal{I}$ and an $\mathcal{I}$wQN but not wQN-space.

Proof.

Take $\mathcal{I} = (\mathrm{Fin} \otimes \mathrm{Fin}) \cap (\emptyset \otimes \mathcal{J})$. We have:
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Additivity

Corollary (from the previous slides)

$\mathtt{non}(\mathcal{I}QN\text{-space}) = \mathfrak{b}$ *for every ideal contained in some* $\mathtt{F}_\sigma$ *ideal.*

Theorem (Das and Chandra)

$\mathtt{add}(\mathcal{I}QN\text{-space}) \geq \mathfrak{b}$ *for every P-ideal.*

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Tall ideals

$\mathcal{I}$ is tall if any infinite set contains an infinite subset from $\mathcal{I}$.

Theorem (Bukovský, Das and Šupina)

For non-tall ideals the notions of $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space) and QN-space (wQN-space) coincide.

Theorem

Let $\mathcal{I}$ be tall. Then any $\mathcal{I}$wQN-space of cardinality $< \mathrm{cov}^*(\mathcal{I})$ is wQN.

$\mathrm{cov}^*(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{X \in [\omega]^\omega} \exists_{A \in \mathcal{A}} |A \cap X| = \omega\right\}$

- $\mathfrak{p} \leq \mathrm{cov}^*(\mathcal{I}) \leq \mathfrak{c}$ for any tall ideal;
- Meza: $\mathrm{cov}^*(\mathrm{conv}) = \mathfrak{c}$, where conv is the ideal on $\mathbb{Q} \cap [0,1]$ generated by sequences in $\mathbb{Q} \cap [0,1]$ convergent in $[0,1]$; conv is $\mathrm{F}_{\sigma\delta\sigma}$;
- Meza: $\mathrm{cov}^*(\mathcal{ED}) = \mathrm{non}(\mathcal{M})$, where $\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by vertical lines and graphs of functions from ω^ω; $\mathcal{ED}$ is F_σ.

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Tall ideals

$\mathcal{I}$ is tall if any infinite set contains an infinite subset from $\mathcal{I}$.

Theorem (Bukovský, Das and Šupina)

For non-tall ideals the notions of $\mathcal{I}$QN-space ($\mathcal{I}$wQN-space) and QN-space (wQN-space) coincide.

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Let $\mathcal{I}$ be tall. Then any $\mathcal{I}$wQN-space of cardinality $< \mathrm{cov}^*(\mathcal{I})$ is wQN.

$$\mathrm{cov}^*(\mathcal{I}) = \min \left\{ |\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \forall_{X \in [\omega]^\omega} \ \exists_{A \in \mathcal{A}} \ |A \cap X| = \omega \right\}$$

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Ideal version of Scheepers' Conjecture

A sequence (U_n) of subsets of a topological space X is an $\mathcal{I}$-γ-cover if $U_n \neq X$ for all n and $\{n : x \notin U_n\} \in \mathcal{I}$ for all $x \in X$. $\mathcal{I}$-Γ is the family of all open $\mathcal{I}$-γ-covers. Moreover, Fin-$\Gamma = \Gamma$.

Conjecture (Scheepers)

FinwQN-space is $S_1(\Gamma, \Gamma)$.

Theorem (Šupina)

If $\mathcal{I}$ is not a weak P-ideal, then there is a perfectly normal $\mathcal{I}$wQN-space which is not $S_1(\Gamma, \mathcal{I}\text{-}\Gamma)$.

Recall that for non-weak P-ideals every topological space is $\mathcal{I}$wQN.

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Ideal version of Scheepers' Conjecture

Consistently, the ideal version of Scheepers' Conjecture does not hold even for some weak P-ideals:

Corollary

If $\mathfrak{b} < \mathfrak{b}_{\mathcal{J}}$ for some $\mathcal{J}$, then there are a weak P-ideal $\mathcal{I}$ and a perfectly normal $\mathcal{I}$wQN-space which is not $S_1(\Gamma, \mathcal{I}\text{-}\Gamma)$.

Proof.

Take $\mathcal{I} = (\mathrm{Fin} \otimes \mathrm{Fin}) \cap (\emptyset \otimes \mathcal{J})$. Then $\mathtt{non}(\mathcal{I}\text{wQN-space}) > \mathfrak{b}$. Šupina proved that $\mathtt{non}(\mathrm{S}_1(\Gamma, \mathcal{I}\text{-}\Gamma)) = \mathfrak{b}_{\mathcal{I}}$. As $\mathrm{Fin} \subseteq \mathcal{I} \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$, $\mathfrak{b} \leq \mathfrak{b}_{\mathcal{I}} \leq \mathfrak{b}_{\mathrm{Fin}\otimes\mathrm{Fin}}$. By a result of Farkas and Soukup, $\mathfrak{b}_{\mathrm{Fin}\otimes\mathrm{Fin}} = \mathfrak{b}$. □

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